Algebraic Lattices and Description Logics

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Abstract

Description logics can be studied from an algebraic point of view cf. [1]. This sheds new light on description logics in several ways. Here we present an algebraic approach to representation of concepts which is based on lattice algebra. We show how to translate this framework into a well known description logic, but argue that during this process one of the key features—the ability to represent conceptual models with an *intensional* element—is lost.

First, the algebraic framework is introduced. It is inspired by the logico-algebraic approach of [2]. Let C be a set of concepts. We have two binary operations on C: conceptual sum $\vee$, and conceptual product $\wedge$. They satisfy the lattice axioms for all $c, d, e \in C$:

$$\begin{aligned} c \vee c &= c, & c \wedge c &= c, \\ c \vee d &= d \vee c, & c \wedge d &= d \wedge c, \\ c \vee (d \vee e) &= (c \vee d) \vee e, & c \wedge (d \wedge e) &= (c \wedge d) \wedge e, \\ c \vee (c \wedge d) &= c, & c \wedge (c \vee d) &= c. \end{aligned}$$

This means that conceptual sum is the least upper bound of its arguments (supremum), and conceptual product is the greatest lower bound of its arguments (infimum). A least concept null and a greatest concept univ can be introduced as boundaries. They satisfy for all $c \in C$:

$$c \wedge \mathsf{null} = \mathsf{null}, \quad c \vee \mathsf{null} = c, \quad c \wedge \mathsf{univ} = c, \quad c \vee \mathsf{univ} = \mathsf{univ}.$$

Furthermore, in order to express various binary relations between the concepts, we introduce a set of attributes A. Attributes are unary operators which satisfy for all $c, d \in C$ and $a \in A$:

$$a(c \vee d) = a(c) \vee a(d), \quad a(c \wedge d) = a(c) \wedge a(d), \quad a(\mathsf{null}) = \mathsf{null}.$$

Concept descriptions are considered to be terms of this algebra. For example, we may have the identities:

$$\begin{aligned} \mathsf{holy\ water} &= \mathsf{water} \wedge \mathit{blessed\_by}(\mathsf{priest}), \\ \mathsf{unhappy\ mother} &= \mathsf{person} \wedge \mathit{has\_child}(\mathsf{sad} \wedge \mathsf{person}), \end{aligned}$$

expressing that holy water is water blessed by a priest, and an unhappy mother is a person who has a child who is sad and a person.

Every model of this algebra is (isomorphic to) a bounded lattice with unary operators $\langle C, \vee, \wedge, \mathsf{univ}, \mathsf{null}, a\rangle_{a\in A}$. So the conceptual models that can be expressed in this framework are those lattices. Note that a model need not be distributive (may not satisfy $c \wedge (d \vee e) = (c \wedge d) \vee (c \wedge e)$).

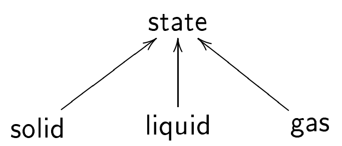
The terms may be given a model-theoretic semantics. This can be realized by a translation into the description logic $\mathcal{ALUE}$. A translation function τ mapping constant concepts into atomic $\mathcal{ALUE}$ concepts, and attributes into atomic roles should satisfy:

$$\begin{aligned} \tau(\mathsf{null}) &= \bot, & \tau(\mathsf{univ}) &= \top, & & \\ \tau(c \vee d) &= \tau(c) \sqcup \tau(d), & \tau(c \wedge d) &= \tau(c) \sqcap \tau(d), & \tau(a(c)) &= \exists\tau(a).(\tau c). \end{aligned}$$

However, this does not give a satisfying account of the semantics, because the algebra admits models that are not distributive, whereas models of description logics like $\mathcal{ALUE}$ are distributive ($2^{\Delta^{\mathcal{I}}}$ is a Boolean algebra).

This could turn out to be a problem, because in some cases we seem to accept conceptual models that are not distributive. Take the following example of (classical) physical states:

$$\mathsf{state} = \mathsf{solid} \vee \mathsf{liquid} \vee \mathsf{gas}.$$



It seems acceptable that the order pictured next to the identity constitutes the entire model for this description (because of lack of space, the least element isn't pictured, but it should be added at the bottom). However, this order cannot be a model of a description logic because it is not distributive. When we accept this model it is because we do not identify a state like solid with the set of entities that are solid. If this was the case, then there would be a distinctive substate like $\mathsf{solid} \vee \mathsf{gas}$ contradicting that there are only three substates. (We *do* admit that there is a set of solids and gasses, but this set should not be identified with a state). Instead, we give solid an *intensional* semantics identifying it with its information content. The upshot is that the lattice algebra admits models which can be characterized as intensional as opposed to description logics.

References

[1] Chris Brink, Katarina Britz, and Renate A. Schmidt. Peirce algebras. *Formal Aspects of Computing*, 6:339–358, 1994.

[2] Jørgen Fischer Nilsson and Jari Palomäki. Towards computing with extensions and intensions of concepts. In P.-J. Charrel et al., editor, *Information Modelling and Knowledge Bases IX*, pages 100–114. IOS Press, 1998.