Reasoning in the $\mathcal{SHOQ}(\mathbf{D_n})$ Description Logic

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Abstract

Description Logics (DLs) are of crucial importance to the development of the Semantic Web, where their role is to provide formal underpinnings and automated reasoning services for Semantic Web ontology languages such as DAML+OIL. In this paper[1] we show how the description logic $\mathcal{SHOQ}(\mathbf{D})$, which has been designed to provide such services, can be extended with n-ary datatype predicates as well as datatype number restrictions, to give $\mathcal{SHOQ}(\mathbf{D_n})$, and we present an algorithm for deciding the satisfiability of $\mathcal{SHOQ}(\mathbf{D_n})$ concepts, along with a proof of its soundness and completeness. The work is motivated by the requirement for n-ary datatype predicates in relation to "real world" properties such as size, weight and duration, in the Semantic Web applications.

1 Introduction

Description Logics (DLs) are of crucial importance to the development of the so-called "Semantic Web" [2], where their role is to provide formal underpinnings and automated reasoning services for Semantic Web ontology languages [3, 10], such as DAML+OIL [6]. Significant effort has already been devoted to the investigation of suitable DLs—in particular, Horrocks and Sattler [7] have presented the $\mathcal{SHOQ}(\mathbf{D})$ DL, along with a sound and complete algorithm for deciding concept satisfiability, a basic reasoning service for DLs and ontologies. A key feature of $\mathcal{SHOQ}(\mathbf{D})$ is that, like DAML+OIL, it supports *datatypes* [1] (e.g., string, integer) as well as the usual abstract concepts (e.g., animal, plant).

$\mathcal{SHOQ}(\mathbf{D})$, however, supports a very restricted form of datatypes, i.e., it can only deal with unary datatype predicates. While this is enough for the *current version* of the DAML+OIL language, it may not be adequate for some Semantic Web applications and for possible future extensions of DAML+OIL. E.g., ontologies used in e-commerce may want to classify different items according to their sizes, and to reason that an item which has *height* less than 5cm and the sum of its *length* and *width* less than 10cm is a kind of item for which no shipping costs are charged. Here "less than 5cm(*height*)" is a unary datatype predicate, and "the sum less than 10cm(*length,width*)" is a binary predicate (see also the according $\mathcal{SHOQ}(\mathbf{D_n})$-concept in Section 2). As shown above, unary predicates are *not enough* in the example, while n-ary datatype predicates are often necessary for "real world" properties, such as size, weight, duration etc., in the ontology applications.

[1] Also available at http://www.cs.man.ac.uk/~panz/Zhilin/download/Paper/Pan-Horrocks-shoqdn-2002.pdf

An approach of extending DL with datatypes was first introduced by Baader and Hanschke [1], who described a datatype ($\mathcal{D}$) extension of the well known $\mathcal{ALC}$ DL. Baader and Hanschke [1] have shown that although the satisfiability of $\mathcal{ALC}(\mathcal{D})$ is decidable, if $\mathcal{ALC}(\mathcal{D})$ is extended with transitive closure of features, the satisfiability problem is undecidable. Lutz [8] proved that reasoning with $\mathcal{ALC}(\mathcal{D})$ and general TBoxes is undecidable. In order to extend *expressive* DLs with concrete domains, Horrocks and Sattler [7] proposed a simplified approach on concrete domain and applied this approach on the $\mathcal{SHOQ}(\mathbf{D})$ DL. Pan [9] investigated the simplifying constraints of $\mathcal{SHOQ}(\mathbf{D})$ w.r.t. datatypes, and showed how these could be relaxed in order to extend $\mathcal{SHOQ}(\mathbf{D})$ with n-ary datatype predicates. We should mention that, similar to Baader and Hanschke [1]'s approach, Haarslev et al. [4] extended the $\mathcal{SHN}$ DL with restricted concert domain $(\mathcal{D})^-$ and gave the $\mathcal{SHN}(\mathcal{D})^-$ DL, which supports n-ary datatype.

In this paper, we extend our work in [9] and add datatype number restrictions to give the $\mathcal{SHOQ}(\mathbf{D_n})$ DL, and present a sound and complete decision procedure for concept satisfiability and subsumption. The rest of the paper is organized as follows. In Section 2, we give the definition of the $\mathcal{SHOQ}(\mathbf{D_n})$ DL. In Section 3, we define a tableau for $\mathcal{SHOQ}(\mathbf{D_n})$. In Section 4, we give an algorithm and its decidability proof. Section 5 is a brief discussion on future works of $\mathcal{SHOQ}(\mathbf{D_n})$.

2 $\mathcal{SHOQ}(\mathbf{D_n})$

Definition 1 $\Delta_\mathbf{D}$is the *datatype domain* covering all concrete datatypes.

Definition 2 Let $\mathbf{C}$, $\mathbf{R} = \mathbf{R}_A \uplus \mathbf{R}_\mathbf{D}$, $\mathbf{I}$ be disjoint sets of concept, abstract and concrete role and individual names. For R and S roles, a *role axiom* is either a role inclusion, which is of the form $R \sqsubseteq S$ for $R, S \in \mathbf{R}_A$ or $R, S \in \mathbf{R}_\mathbf{D}$, or a transitivity axiom, which is of the form $\mathsf{Trans}(R)$ for $R \in \mathbf{R}_A$. A *role box* $\mathcal{R}$ is a finite set of role axioms. A role R is called *simple* if, for $\underline{\circledast}$ the transitive reflexive closure of $\sqsubseteq$ on $\mathcal{R}$ and for each role S, $S \underline{\circledast} R$ implies $\mathsf{Trans}(S) \notin \mathcal{R}$.

The set of concept terms of $\mathcal{SHOQ}(\mathbf{D_n})$ is inductively defined. As a starting point of the induction, any element of $\mathbf{C}$ is a concept term (atomic terms). Now let C and D be concept terms, o be an individual, R be a abstract role name, $T_1, \ldots, T_n$ be concrete role names, P be an n-ary predicate name. Then the following expressions are also concept terms:

1. $\top$ (universal concept) and $\top_\mathtt{D}$ (universal datatype),
2. $C \sqcup D$ (disjunction), $C \sqcap D$ (conjunction), $\neg C$ (negation), and $\{o\}$ (nominals),
3. $\exists R.C$ (exists-in restriction) and $\forall R.C$ (value restriction),
4. $\geqslant mR.C$ (atleast restriction) and $\leqslant mR.C$ (atmost restriction),
5. $\exists T_1, \cdots, T_n.P_n$ (datatype exists) and $\forall T_1, \cdots, T_n.P_n$ (datatype value),
6. $\geqslant mT_1, \ldots, T_n.P_n$ (datatype atleast) and $\leqslant mT_1, \ldots, T_n.P_n$ (datatype atmost),
7. $\geqslant mT$, $\leqslant mT$ (number restrictions on concrete roles).

$\mathcal{SHOQ}(\mathbf{D_n})$ extends $\mathcal{SHOQ}(\mathbf{D})$ by supporting n-ary datatype predicates P_n, the interpretation $P_n^\mathbf{D}$ of which is

$$P_n^\mathbf{D} \subseteq d_1^\mathbf{D} \times \cdots \times d_n^\mathbf{D} \subseteq \Delta_\mathbf{D}^n$$

where $d_i^\mathbf{D} \in \Delta_\mathbf{D}$ are datatypes. The interpretation of the projection of P_n is defined as

$$P_n(i)^\mathbf{D} \subseteq d_i^\mathbf{D} \subseteq \Delta_\mathbf{D}$$

Construct Name	Syntax	Semantics
universal datatype	$\top_{\mathtt{D}}$	$\top_{\mathtt{D}}^{\mathbf{D}} = \Delta_{\mathbf{D}}$
datatype predicate	P_n	$P_n^{\mathbf{D}} \subseteq d_1^{\mathbf{D}} \times \cdots \times d_n^{\mathbf{D}} \subseteq \Delta_{\mathbf{D}}^n$
datatype exists	$\exists T_1, \cdots, T_n.P_n$	$(\exists T_1, \cdots, T_n.P_n)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y_1 \cdots y_n.$ $\langle x, y_1\rangle \in T_1^{\mathcal{I}} \wedge \cdots \wedge \langle x, y_n\rangle \in T_n^{\mathcal{I}} \wedge \langle y_1, \cdots y_n\rangle \in P_n^{\mathbf{D}}\}$
datatype value	$\forall T_1, \cdots, T_n.P_n$	$(\forall T_1, \cdots, T_n.P_n)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y_1 \cdots y_n.$ $\langle x, y_1\rangle \in T_1^{\mathcal{I}} \wedge \cdots \wedge \langle x, y_n\rangle \in T_n^{\mathcal{I}} \rightarrow \langle y_1, \cdots y_n\rangle \in P_n^{\mathbf{D}}\}$
datatype atleast	$\geqslant mT_1, \ldots, T_n.P_n$	$(\geqslant mT_1, \ldots, T_n.P_n)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \sharp\{\langle y_1 \cdots y_n\rangle \mid$ $\langle x, y_1\rangle \in T_1^{\mathcal{I}} \wedge \cdots \wedge \langle x, y_n\rangle \in T_n^{\mathcal{I}} \wedge \langle y_1, \cdots y_n\rangle \in P_n^{\mathbf{D}}\} \geq m\}$
datatype atmost	$\leqslant mT_1, \ldots, T_n.P_n$	$(\leqslant mT_1, \ldots, T_n.P_n)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \sharp\{\langle y_1 \cdots y_n\rangle \mid$ $\langle x, y_1\rangle \in T_1^{\mathcal{I}} \wedge \cdots \wedge \langle x, y_n\rangle \in T_n^{\mathcal{I}} \wedge \langle y_1, \cdots y_n\rangle \in P_n^{\mathbf{D}}\} \leq m\}$
concrete role atleast	$\geqslant mT$	$(\geqslant mT)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \in \Delta_{\mathbf{D}} \mid \langle x, y\rangle \in T\} \geq m\}$
concrete role atmost	$\leqslant mT$	$(\leqslant mT)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \in \Delta_{\mathbf{D}} \mid \langle x, y\rangle \in T\} \leq m\}$

Figure 1: Datatype constructs in $\mathcal{SHOQ}(\mathbf{D_n})$

and note that when we say

$$\langle y_1, \ldots, y_n\rangle \in P_n^{\mathbf{D}}$$

we mean: (i) $y_i \in P_n(i)^{\mathbf{D}}$ for $1 \leq i \leq n$, and (ii) $y_1, \ldots, y_n$ satisfy datatype predicate P_n. The interpretations of other datatype constructs are listed in Figure 1. Note that concrete role atleast (atmost) is only a special form of datatype atleast (atmost, respectively) where $n = 1$ and $P_n = \top_{\mathtt{D}}$.

To illustrate the use of $\mathcal{SHOQ}(\mathbf{D_n})$-concept, let's go back to the example we used in Section 1. Items with height less than 5cm, and the sum of their length and width less that 10cm can be defined as a $\mathcal{SHOQ}(\mathbf{D_n})$-concept

$$\mathit{item} \sqcap = 1\mathit{height}.<_{\mathtt{5cm}} \sqcap = 1\mathit{length} \sqcap = 1\mathit{width} \ \sqcap \forall \mathit{length}, \mathit{width}.\mathtt{sum} <_{\mathtt{10cm}}$$

where "=1" is a shortcut for "$\leqslant 1 \sqcap \geqslant 1$", and *height*, *length* and *width* are concrete roles, $<_{\mathtt{5cm}}$ is a unary datatype predicate and $\mathtt{sum} <_{\mathtt{10cm}}$ is a binary predicate. Note that $<_{\mathtt{5cm}}$ and $\mathtt{sum} <_{\mathtt{10cm}}$ are datatype predicates, rather than datatype number restrictions.

Datatypes and predicates are considered to be already sufficiently structured by the *type system*, which may includes a derivation mechanism and built-in ordering relations, so that it can be used to define datatypes d and predicates P_n, as well as negation of predicates $\neg P_n$, it can be used to check if a tuple of values $t_1, \ldots, t_n$ satisfy a predicate P_n, if a set of tuples of values satisfy a set of predicates simultaneously etc. With the type system, we can deal with an arbitrary conforming set of datatypes and predicates without compromising the compactness of the concept language or the soundness and completeness of our decision procedure [7].

3 A Tableau for $\mathcal{SHOQ}(\mathbf{D_n})$

In this section, we define a tableau for $\mathcal{SHOQ}(\mathbf{D_n})$. For ease of presentation, we assume all concepts to be in *negation normal form* (NNF). We use $\sim C$ to denote the NNF of $\neg C$. Moreover, for a concept D, we use $\mathsf{cl}(D)$ to denote the set of all sub-concepts of D, the NNF of these sub-concepts, and the (possibly negated) datatypes occurring in these (NNF of) sub-concepts.

Definition 3 If D is a $\mathcal{SHOQ}(\mathbf{D_n})$-concept in NNF, $\mathcal{R}$ a role box, and $\mathbf{R}_A^D$, $\mathbf{R}_{\mathbf{D}}^D$ are the sets of abstract and concrete roles occurring in D or $\mathcal{R}$, a tableau $\mathcal{T}$ for D w.r.t. $\mathcal{R}$ is defined as a quadruple $(\mathbf{S}, \mathcal{L}, \mathcal{E}_A, \mathcal{E}_{\mathbf{D}})$ such that: $\mathbf{S}$ is a set of individuals, $\mathcal{L} : \mathbf{S} \rightarrow 2^{\mathsf{cl}(D)}$ maps each individual to a set of concepts which is a subset of $\mathsf{cl}(D)$, $\mathcal{E}_A : \mathbf{R}_A^D \rightarrow 2^{\mathbf{S}\times\mathbf{S}}$ maps each abstract role in $\mathbf{R}_A^D$ to a set of pairs of individuals, $\mathcal{E}_\mathbf{D} : \mathbf{R}_\mathbf{D}^D \to 2^{\mathbf{S}\times\Delta_\mathbf{D}}$ maps each concrete role in $\mathbf{R}_\mathbf{D}^D$ to a set of pairs of individuals and concrete values, and there is some individual $s \in \mathbf{S}$ such that $D \in \mathcal{L}(s)$. For all $s, t \in \mathbf{S}$, $C, C_1, C_2 \in cl(D)$, $R, S \in \mathbf{R}_A^D$, $T, T', T_1, \ldots, T_n \in \mathbf{R}_\mathbf{D}^D$, n-ary predicate P_n and

$$\begin{array}{rcl} S^{\mathcal{T}}(s, C) & := & \{t \in \mathbf{S} \mid \langle s, t\rangle \in \mathcal{E}_A(S) \text{ and } C \in \mathcal{L}(t)\}, \\ T_1T_2\ldots T_n^{\mathcal{T}}(s, P_n) & := & \{\langle y_1, \ldots, y_n\rangle \in P_n^\mathbf{D} \mid \langle s, y_1\rangle \in \mathcal{E}_\mathbf{D}(T_1), \ldots, \langle s, y_n\rangle \in \mathcal{E}_\mathbf{D}(T_n)\}, \\ DC^{\mathcal{T}}(s, T_1, \ldots, T_n, y_1, \ldots, y_n, P_n) & := & \begin{cases} true & \text{if } \langle s, y_i\rangle \in \mathcal{E}_\mathbf{D}(T_n)(1 \le i \le n) \text{ and} \\ & \langle y_1, \ldots, y_n\rangle \in P_n^\mathbf{D} \\ false & \text{otherwise} \end{cases} \end{array}$$

it holds that:

(P1) if $C \in \mathcal{L}(s)$, then $\neg C \notin \mathcal{L}(s)$,

(P2) if $C_1 \sqcap C_2 \in \mathcal{L}(s)$, then $C_1 \in \mathcal{L}(s)$ and $C_2 \in \mathcal{L}(s)$,

(P3) if $C_1 \sqcup C_2 \in \mathcal{L}(s)$, then $C_1 \in \mathcal{L}(s)$ or $C_2 \in \mathcal{L}(s)$,

(P4) if $\langle s, t\rangle \in \mathcal{E}_A(R)$ and $R \mathrel{\underline{\circledast}} S$, then $\langle s, t\rangle \in \mathcal{E}_A(S)$,
if $\langle s, t\rangle \in \mathcal{E}_\mathbf{D}(T)$ and $T \mathrel{\underline{\circledast}} T'$, then $\langle s, t\rangle \in \mathcal{E}_\mathbf{D}(T')$,

(P5) if $\forall R.C \in \mathcal{L}(s)$ and $\langle s, t\rangle \in \mathcal{E}_A(R)$, then $C \in \mathcal{L}(t)$,

(P6) if $\exists R.C \in \mathcal{L}(s)$, then there is some $t \in \mathbf{S}$ such that $\langle s, t\rangle \in \mathcal{E}_A(R)$ and $C \in \mathcal{L}(t)$,

(P7) if $\forall S.C \in \mathcal{L}(s)$ and $\langle s, t\rangle \in \mathcal{E}_A(R)$ for some $R \mathrel{\underline{\circledast}} S$ with $\mathsf{Trans}(R)$, then $\forall R.C \in \mathcal{L}(t)$,

(P8) if $\geqslant nS.C \in \mathcal{L}(s)$, then $\sharp S^T(s, C) \geqslant n$,

(P9) if $\leqslant nS.C \in \mathcal{L}(s)$, then $\sharp S^T(s, C) \leqslant n$,

(P10) if $\{\leqslant nS.C, \geqslant nS.C\} \cap \mathcal{L}(s) \neq \emptyset$ and $\langle s, t\rangle \in \mathcal{E}_A(S)$, then $\{C, {\sim}C\} \cap \mathcal{L}(t) \neq \emptyset$,

(P11) if $\{o\} \in \mathcal{L}(s) \cap \mathcal{L}(t)$, then $s = t$,

(P12) if $\forall T_1, \cdots, T_n.P_n \in \mathcal{L}(s)$ and $\langle s, t_1\rangle \in \mathcal{E}_\mathbf{D}(T_1), \cdots, \langle s, t_n\rangle \in \mathcal{E}_\mathbf{D}(T_n)$, then $DC^{\mathcal{T}}(s, T_1, \ldots, T_n, t_1, \ldots, t_n, P_n) = true$,

(P13) if $\exists T_1, \cdots, T_n.P_n \in \mathcal{L}(s)$, then there is some $t_1, \cdots, t_n \in \Delta_\mathbf{D}$ such that $\langle s, t_1\rangle \in \mathcal{E}_\mathbf{D}(T_1), \cdots, \langle s, t_n\rangle \in \mathcal{E}_\mathbf{D}(T_n)$, $DC^{\mathcal{T}}(s, T_1, \ldots, T_n, t_1, \ldots, t_n, P_n) = true$,

(P14) if $\geqslant mT_1, \ldots, T_n.P_n \in \mathcal{L}(s)$, then $\sharp T_1T_2\ldots T_n^T(s, P_n) \geqslant m$,

(P15) if $\leqslant mT_1, \ldots, T_n.P_n \in \mathcal{L}(s)$, then $\sharp T_1T_2\ldots T_n^T(s, P_n) \leqslant m$,

(P16) if $\{\leqslant mT_1, \ldots, T_n.P_n, \geqslant mT_1, \ldots, T_n.P_n\} \cap \mathcal{L}(s) \neq \emptyset$ and $\langle s, t_1\rangle \in \mathcal{E}_\mathbf{D}(T_1)$, …, $\langle s, t_n\rangle \in \mathcal{E}_\mathbf{D}(T_n)$, then for $1 \le i \le n$, we have either $DC^{\mathcal{T}}(s, T_1, \ldots, T_n, t_1, \ldots, t_n, P_n) = true$, or $DC^{\mathcal{T}}(s, T_1, \ldots, T_n, t_1, \ldots, t_n, \neg P_n) = true$.

Lemma 4 A $\mathcal{SHOQ}(\mathbf{D_n})$-concept D in NNF is satisfiable w.r.t. a role box $\mathcal{R}$ iff D has a tableau w.r.t. $\mathcal{R}$.

Proof: For the *if* direction, if $\mathcal{T} = (\mathbf{S}, \mathcal{L}, \mathcal{E}_A, \mathcal{E}_\mathbf{D})$ is a tableau for D, a model $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ of D can be defined as: $\Delta^\mathcal{I} = \mathbf{S}$, $\mathsf{CN}^\mathcal{I} = \{s \mid \mathsf{CN} \in \mathcal{L}(s)\}$ for all concept names CN in $\mathtt{cl}(D)$, if $R \in \mathbf{R}_+$, $\mathbf{R}_A^\mathcal{I} = \mathcal{E}_A(R)^+$, otherwise $\mathbf{R}_A^\mathcal{I} = \mathcal{E}_A(R) \cup \bigcup_{P \mathrel{\underline{\circledast}} R, P \neq R} P^\mathcal{I}$, $\mathbf{R}_\mathbf{D}^\mathcal{I} = \mathcal{E}_D(R)$, where $\mathcal{E}_A(R)^+$ denotes the transitive closure of $\mathcal{E}_A(R)$. $D^\mathcal{I} \neq \emptyset$ because $s_0 \in D^\mathcal{I}$. Here we only concentrate on (P14) to (P15); the remainder is similar to the proofs found in [9][2].

[2]Note that in this paper, we mainly focus on the proof of the number restriction on concrete roles, the remainder is similar to the proofs found in [9].

1. $E = \geqslant mT_1, \ldots, T_n.P_n$. According to (P14), $E \in \mathcal{L}(s)$ implies that $\sharp T_1T_2 \ldots T_n^{\mathcal{T}}(s, P_n) \geqslant m$. By the definition of $T_1T_2 \ldots T_n^{\mathcal{T}}(s, P_n)$, we have $s \in \{x \in \Delta^{\mathcal{I}} \mid \sharp\{\langle t_1, \ldots, t_n\rangle \mid \langle x, t_1\rangle \in \mathcal{E}_{\mathbf{D}}(T_1) \wedge \ldots \wedge \langle x, t_n\rangle \in \mathcal{E}_{\mathbf{D}}(T_n) \wedge \langle t_1, \ldots, t_n\rangle \in P_n^{\mathbf{D}}\} \geq m\}$, Since $\mathcal{E}_{\mathbf{D}}(T_i) = T_i^{\mathcal{I}}$, we have $s \in (\geqslant mT_1, \ldots, T_n.P_n)^{\mathcal{I}}$. Similarly, if $E = \leqslant mT_1, \ldots, T_n.P_n$, we have $s \in (\leqslant mT_1, \ldots, T_n.P_n)^{\mathcal{I}}$.

For the converse, if $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a model of D, then a tableau $\mathcal{T} = (\mathbf{S}, \mathcal{L}, \mathcal{E}_A, \mathcal{E}_{\mathbf{D}})$ for D can be defined as: $S = \Delta^{\mathcal{I}}$, $\mathcal{E}_A(R) = R_A^{\mathcal{I}}$, $\mathcal{E}_{\mathbf{D}}(R) = R_D^{\mathcal{I}}$, $\mathcal{L}(s) = \{C \in \mathtt{cl}(D) \mid s \in C^{\mathcal{I}}\}$. It only remains to demonstrate that T is a tableau for D: $\mathcal{T}$ satisfies (P14) to (P16) as a direct consequence of the semantics of datatype constructs. ∎

4 A Tableau Algorithm for $\mathcal{SHOQ}(\mathbf{D_n})$

Form Lemma 4, an algorithm which constructs a tableau for a $\mathcal{SHOQ}(\mathbf{D_n})$-concept D can be used as a decision procedure for the satisfiability of D with respect to a role box $\mathcal{R}$.

Definition 5 Let $\mathcal{R}$ be a role box, D a $\mathcal{SHOQ}(\mathbf{D_n})$-concept in NNF, $\mathbf{R}_A^D$ the set of abstract roles occurring in D or $\mathcal{R}$, and $\mathbf{I}^D$ the set of nominal occurring in D. A tableaux algorithm works on a *completion forest* for D w.r.t. $\mathcal{R}$, which is a set of trees $\mathtt{F}$. Each node x of the forest is labelled with a set

$$\mathcal{L}(x) \subseteq \mathtt{cl}(D) \cup \{\uparrow (R, \{o\}) \mid R \in \mathbf{R}_A^D \text{ and } \{o\} \in \mathbf{I}^D\},$$

and each edge $\langle x, y\rangle$ is labelled with a set of role names $\mathcal{L}(\langle x, y\rangle)$ containing roles occurring in $\mathrm{cl}(D)$ or $\mathcal{R}$. Concrete values are represented by concrete nodes, which are always leaves of $\mathtt{F}$. Additionally, we keep track of inequalities between nodes of the tree with a symmetric binary relation $\neq$ between the nodes of $\mathtt{F}$. For each $\{o\} \in \mathbf{I}^D$ there is a *distinguished* node $x_{\{o\}}$ in $\mathtt{F}$ such that $\{o\} \in \mathcal{L}(x)$. The algorithm expands the forest either by extending $\mathcal{L}(x)$ for some node x or by adding new leaf nodes.

Given a completion forest, a node y is called an *R-successor* of a node x if, for some R' with $R' \underline{\circledast} R$, either y is a successor of x and $R' \in \mathcal{L}(\langle x, y\rangle)$, or $\uparrow (R', \{o\}) \in \mathcal{L}(x)$ and $y = x_{\{o\}}$. Ancestors and roots are defined as usual. For an abstract role S and a node x in $\mathtt{F}$ we define $S^{\mathtt{F}}(x, C)$ by

$$S^{\mathtt{F}}(x, C) := \{y \mid y \text{ is an } S\text{-successor of x and } C \in \mathcal{L}(y)\}.$$

Given a completion forest, concrete nodes $t_1, \ldots, t_n$ are called *$T_1T_2 \ldots T_n$-successors* of a node x if, for some concrete roles $T_1', \ldots, T_n'$ with $T_i' \underline{\circledast} T_i$, $t_1, \ldots, t_n$ are successors of x and $T_i' \in \mathcal{L}(\langle x, t_i\rangle), 1 \leq i \leq n$. For a node x, its $T_1T_2 \ldots T_n$-successors $\langle t_1, \ldots, t_n\rangle$, n-ary datatype predicate P_n, we define a set $DC^{\mathtt{F}}$ by

$$DC^F = \{< DCelement >\}$$

where $DC^{\mathtt{F}}$ is a set of $DCelements$, which have the form

$$< DCelement >= \{x, \langle T_1, \ldots, T_n\rangle, \langle t_1, \ldots, t_n\rangle, P_n\}$$

$DC^{\mathtt{F}}$ is initialised as an empty set. $DC^{\mathtt{F}}$is is *satisfied* iff. there exists $value : N_C \to \Delta_{\mathbf{D}}$, where N_C is the set of all concrete nodes, s.t. for all $\{x, \langle T_1, \ldots, T_n\rangle, \langle t_1, \ldots, t_n\rangle, P_n\} \in DC^{\mathtt{F}}$, $\langle value(t_1), \ldots, value(t_n)\rangle \in P_n^{\mathbf{D}}$ are true. In order to retrieve the set of all the $T_1T_2 \ldots T_n$-successors of x, which satisfy a certain predicate P_n, we define $DCSuccessors^{\mathtt{F}}(x, P_n)$ by

$$DCSuccessors^{\mathtt{F}}(x, T_1, \ldots, T_n, P_n) := \{\langle t_1, \ldots, t_n\rangle \mid \{x, \langle T_1, \ldots, T_n\rangle, \langle t_1, \ldots, t_n\rangle, P_n\} \in DC^{\mathtt{F}}\}$$

In order to retrieve the set of datatype predicates, which are satisfied by $T_1T_2 \ldots T_n$-successors $t_1, \ldots, t_n$ of x, we define $DCPredicates^{\mathtt{F}}(x, T_1, \ldots, T_n, t_1, \ldots, t_n)$by

$$DCPredicates^{\mathtt{F}}(x, T_1, \ldots, T_n, t_1, \ldots, t_n) := \{P_n \mid \{x, \langle T_1, \ldots, T_n\rangle, \langle t_1, \ldots, t_n\rangle, P_n\} \in DC^{\mathtt{F}}\}$$

$\forall_P$-rule:	if 1.$\forall T_1, \cdots, T_n.P_n \in \mathcal{L}(x)$, x is not blocked, and 2.there are $T_1 T_2 \ldots T_n$-successors $\langle t_1, \ldots, t_n \rangle$ of x with $P_n \notin DCPredicates^{\mathtt{F}}(x, T_1, \ldots, T_n, t_1, \ldots, t_n)$, then $DC^{\mathtt{F}} \longrightarrow DC^{\mathtt{F}} \cup \{x, \langle T_1, \ldots, T_n \rangle, \langle t_1, \ldots, t_n \rangle, P_n\}$.
$\exists_P$-rule:	if 1.$\exists T_1, \cdots, T_n.P_n \in \mathcal{L}(x)$, x is not blocked, and 2.there are no $T_1 T_2 \ldots T_n$-successors $\langle t_1, \ldots, t_n \rangle$ of x, with $P_n \in DCPredicates^{\mathtt{F}}(x, T_1, \ldots, T_n, t_1, \ldots, t_n)$, then 1. create $T_1 T_2 \ldots T_n$-successors $\langle t_1, \cdots, t_n \rangle$ with $\mathcal{L}(\langle x, t_i \rangle) = \{Ti\}$ for $1 \leq i \leq n$ and 2. $DC^{\mathtt{F}} \longrightarrow DC^{\mathtt{F}} \cup \{x, \langle T_1, \ldots, T_n \rangle, \langle t_1, \ldots, t_n \rangle, P_n\}$.
$\geqslant_P$-rule:	if 1.$\geqslant m T_1, \cdots, T_n.P_n \in \mathcal{L}(x)$, x is not blocked, and 2.$\sharp DCSuccessors^{\mathtt{F}}(x, T_1, \ldots, T_n, P_n) < m$, then 1. create m $T_1 T_2 \ldots T_n$-successors $\langle t_{11}, \ldots, t_{1n} \rangle, \cdots, \langle t_{m1}, \ldots, t_{mn} \rangle$, with $\mathcal{L}(\langle x, t_{ji} \rangle) \longrightarrow \{Ti\}$, and 2. $DC^{\mathtt{F}} \longrightarrow DC^{\mathtt{F}} \cup \{x, \langle T_1, \ldots, T_n \rangle, \langle t_{j1}, \ldots, t_{jn} \rangle, P_n\}$ and 3. set $\langle t_{j1}, \ldots, t_{jn} \rangle \dot{\neq} \langle t_{k1}, \ldots, t_{kn} \rangle$, for all $1 \leq i \leq n$, $1 \leq j < k \leq m$.
$\leqslant_P$-rule:	if 1.$\leqslant m T_1, \cdots, T_n.P_n \in \mathcal{L}(x)$, x is not blocked, and 2.$\sharp DCSuccessors^{\mathtt{F}}(x, T_1, \ldots, T_n, P_n) > m$ and 3.there exist $j \neq k$, s.t. $\langle t_{j1}, \ldots, t_{jn} \rangle, \langle t_{k1}, \ldots, t_{kn} \rangle \in DCSuccessors^{\mathtt{F}}(x,$ $T_1, \ldots, T_n, P_n)$ but not $\langle t_{j1}, \ldots, t_{jn} \rangle \dot{\neq} \langle t_{k1}, \ldots, t_{kn} \rangle, 1 \leq j < k \leq m+1$, then 1. $\mathcal{L}(\langle x, t_{ki} \rangle) \longrightarrow \mathcal{L}(\langle x, t_{ki} \rangle) \cup \mathcal{L}(\langle x, t_{ji} \rangle)$, and 2. $DC^{\mathtt{F}} \longrightarrow DC^{\mathtt{F}}[\langle t_{j1}, \ldots, t_{jn} \rangle / \langle t_{k1}, \ldots, t_{kn} \rangle] \mid_{x, T_1, \ldots, T_n, P_n}$, and 3. add $u \dot{\neq} \langle t_{k1}, \ldots, t_{kn} \rangle$ for each tuple u with $u \dot{\neq} \langle t_{j1}, \ldots, t_{jn} \rangle$, and 4. remove all t_{ji} where t_{ji} isn't in any tuples of$DCSuccessors^{\mathtt{F}}(x, *, *)$ and remove all edges leading to these t_{ji} from $\mathtt{F}$.
$choose_P$-rule:	if 1.$\{\leqslant m T_1, \cdots, T_n.P_n, \geqslant m T_1, \cdots, T_n.P_n\} \cap \mathcal{L}(x) \neq \emptyset$, x is not blocked, and 2.$\langle t_1, \ldots, t_n \rangle$ are $T_1 T_2 \ldots T_n$-successors of x, and then either $DC^{\mathtt{F}} \longrightarrow DC^{\mathtt{F}} \cup \{x, \langle T_1, \ldots, T_n \rangle, \langle t_1, \ldots, t_n \rangle, P_n\}$, or $DC^{\mathtt{F}} \longrightarrow DC^{\mathtt{F}} \cup \{x, \langle T_1, \ldots, T_n \rangle, \langle t_1, \ldots, t_n \rangle, \neg P_n\}$.

Figure 2: The Tableaux Expansion Rules for $\mathcal{SHOQ}(\mathbf{D_n})$ (I)

Note that we can use * as parameter in $DCSuccessors^{\mathtt{F}}$ and $DCPredicates^{\mathtt{F}}$, e.g. $DCSuccess-$ $ors^{\mathtt{F}}(x, *, *)$ means all the concrete successors of node x.

A node x is *directly blocked* if none of its ancestors are blocked, and it has an ancestor x' that is not distinguished such that $\mathcal{L}(x) \subseteq \mathcal{L}(x')$. We call x' blocks x. A node is *blocked* if it is directly blocks or if its predecessor is blocked.

If $\{o_1\}, \cdots, \{o_l\}$ are all individuals occurring in D, the algorithm initialises the completion forest $\mathtt{F}$ to contain $l+1$ root nodes $x_0, x_{\{o_1\}}, \cdots, x_{\{o_l\}}$ with $\mathcal{L}(x_0) = \{D\}$ and $\mathcal{L}(x_{\{o_i\}}) = \{\{o_i\}\}$. The inequality relation $\dot{\neq}$ is initialised with the empty relation. $\mathtt{F}$ is then expended by repeatedly applying the *expansion rules*, listed in Figure 2[3], stopping if a *clash* occurs in one of its nodes.

For a node x, $\mathcal{L}(x)$ is said to contain a *clash* if:

1. for some concept name $A \in N_C$, $\{A, \neg A\} \subseteq \mathcal{L}(x)$, or
2. for some role S, $\leqslant S.C \in \mathcal{L}(x)$ and there are $n+1$ S-successors $y_0, \cdots, y_n$ of x with $C \in \mathcal{L}(y_i)$ for each $0 \leq i \leq n$ and $y_i \neq y_j$ for each $0 \leq i < j \leq n$, or
3. $DC^{\mathtt{F}}$ isn't satisfied;
4. for some concrete roles $T_1, \ldots, T_n$, n-ary datatype predicate P_n, $\leqslant m T_1, \ldots, T_n.P_n \in \mathcal{L}(x)$, we have $\sharp T_1 T_2 \ldots T_n^{\mathtt{F}}(x, P_n) \geq m+1$, or

[3]Figure 2 only lists the rules about datatypes, other rules can be found in [9].

5. for some $\{o\} \in \mathcal{L}(x), x \dot{\neq} x_{\{o\}}$.

The completion forest is *complete* when, for some node x, $\mathcal{L}(x)$ contains a clash, or when none of the expansion rules is applicable. If the expansion rules can be applied in such a way that they yield a complete, clash-free completion forest, then the algorithm returns "D is *satisfiable* w.r.t. $\mathcal{R}$", and "D is *unsatisfiable* w.r.t. $\mathcal{R}$" otherwise.

Lemma 6 (Termination) When started with a $\mathcal{SHOQ}(\mathbf{D_n})$-concept D in NNF, the tableau algorithm terminates.

Proof: Let $d = |cl(D)|$, $k = |\mathbf{R}_A^D|$, n_{max} the maximal number in atleast number restrictions as well as datatype atleast, and $\ell = |\mathcal{I}^D|$. Here we mainly concentrate on rules about number restriction on concrete roles. Termination is a consequence of the following properties of the expansion rules:

1. Each rule but the $\leqslant$-, $\leqslant_P$- or the $\mathbf{O}$-rule strictly extends the completion forest, by extending node labels or adding nodes, while removing neither nodes nor elements from node.
2. New nodes are only generated by the $\exists$-, $\exists_P$-, $\geqslant$-rule or the $\geqslant_P$-rule as successors of a node x for concepts of the form $\exists R.C$, $\exists T_1, \cdots, T_n.P_n$, $\geqslant nS.C$ and $\geqslant mT_1, \cdots, T_n.P_n$ in $\mathcal{L}(x)$. For a node x, each of these concepts can trigger the generation of successors at most once—even though the node(s) generated was later removed by either the $\leqslant$-, $\leqslant_P$- or the $\mathbf{O}$-rule. For the $\geqslant_P$-rule: If $T_1T_2 \ldots T_n$-successors $\langle t_{11}, \cdots, t_{1n}\rangle, \cdots, \langle t_{m1}, \cdots, t_{mn}\rangle$ were generated by an application of the $\geqslant_P$-rule for a concept $(\geqslant mT_1, \cdots, T_n.P_n)$, then $\langle t_{j1}, \cdots, t_{jn}\rangle \dot{\neq} \langle t_{k1}, \cdots, t_{kn}\rangle$ holds for all $1 \leq i \leq n$ and $1 \leq j < k \leq m$. This implies there will always be m $T_1T_2 \ldots T_n$-successors $\langle t_{11}, \cdots, t_{1n}\rangle, \cdots, \langle t_{m1}, \cdots, t_{mn}\rangle$ of x with $P_n(i) \in \mathcal{L}(t_i)$ and $\langle t_{j1}, \cdots, t_{jn}\rangle \dot{\neq} \langle t_{k1}, \cdots, t_{kn}\rangle$ holds for all $1 \leq i \leq n$ and $1 \leq j < k \leq m$, since the $\leqslant$-, $\mathbf{O}$- and $\leqslant_P$-rule can never merge them, and, whenever an application of the $\leqslant_P$-rule sets some $\mathcal{L}(t_{ji})$ to $\emptyset$, then there will be some $T_1T_2 \ldots T_n$-successors $\langle t_{k1}, \cdots, t_{kn}\rangle$ of x with $P_n(i) \in \mathcal{L}(t_{ki})$ and $\langle t_{k1}, \cdots, t_{kn}\rangle$ "inherits" all inequalities from $\langle t_{j1}, \cdots, t_{jn}\rangle$. Hence the out-degree of the forest is bounded by $d \cdot n_{max}$.
3. Nodes are labelled with subsets of $cl(D) \cup \{\uparrow(R, \{o\}) \mid R \in \mathbf{R}_A^D$ and $\{o\} \in \mathcal{I}^D\}$, and the concrete value nodes are always leaves, so there are at most $2^{d+k\ell}$ different node labellings. Therefore, if a path p is of length at least $2^{d+k\ell}$, then, from the blocking condition above, there are two nodes x, y on p such that x is directly blocked by y. Hence paths are of length at most $2^{d+k\ell}$. ∎

Lemma 7 (Soundness) If the expansion rules can be applied to a $\mathcal{SHOQ}(\mathbf{D_n})$-concept D in NNF and a role box $\mathcal{R}$ such that they yield a complete and clash-free completion forest, then D has a tableau w.r.t. $\mathcal{R}$.

Proof: Let $\mathtt{F}$ be the complete and clash-free completion forest constructed by the tableaux algorithm for D. To cope with cycle, an individual in $\mathbf{S}$ corresponds to a *path* in $\mathtt{F}$. Due to qualifying number restrictions, we must distinguish different nodes that are blocked by the same node. We refer the readers to [9] for the definitions of path and related concepts. We can define a tableau $\mathcal{T} = (\mathbf{S}, \mathcal{L}, \mathcal{E}_A, \mathcal{E}_\mathbf{D})$ with: $\mathbf{S} = \mathtt{Paths}(\mathtt{F})$, $\mathcal{L}(p) = \mathcal{L}(\mathtt{Tail}(p))$, $\mathcal{E}_A(R_A) = \{\langle p, q\rangle \in \mathbf{S} \times \mathbf{S} \mid q = [p|(x, x')]$ *and* x' is an R_A-successor of $\mathtt{Tail}(p),\}$ $\mathcal{E}_\mathbf{D}(R_\mathbf{D}) = \{\langle p, t\rangle \in \mathbf{S} \times \Delta_\mathbf{D} \mid t$ is an $R_\mathbf{D}$-successor of $\mathtt{Tail}(p)\}$.

We have to show that $\mathcal{T}$ satisfies (P14) to (P17) from Definition 3.

- (P14): Assume $\geqslant mT_1, \ldots, T_n.P_n \in \mathcal{L}(p)$. This implies that in $\mathtt{F}$ there exist m $T_1T_2 \ldots T_n$-successors $\langle t_{11}, \ldots, t_{1n}\rangle$, $\ldots$, $\langle t_{m1}, \ldots, t_{mn}\rangle$ of $\mathtt{Tail}(p)$ and $P_n(i) \in \mathcal{L}(t_{ji})$ for all $1 \leq i \leq n, 1 \leq j \leq m$. We claim that , for each of these concrete nodes, according to the construction of $\mathcal{E}_\mathbf{D}$ above, we have $\langle p, t_{ji}\rangle \in \mathcal{E}_D(T_i)$, and $\langle t_{j1}, \cdots, t_{jn}\rangle \dot{\neq} \langle t_{k1}, \cdots, t_{kn}\rangle$ and $\{p, \langle T_1, \ldots, T_n\rangle, \langle t_{j1}, \ldots, t_{jn}\rangle, P_n\} \in DC^{\mathtt{F}}$ for all $1 \leq i \leq n$ and $1 \leq j < k \leq m$ (otherwise, $\geqslant_P$-rule was still applicable). According to the definition of $DC^{\mathtt{F}}$and $T_1T_2 \ldots T_n^{\mathcal{T}}(p, P_n)$, this implies $\sharp T_1T_2 \ldots T_n^{\mathcal{T}}(p, P_n) \geqslant m$.

- (P15): Assume (P15) doesn't hold. Hence there is some $p \in \mathbf{S}$ with $(\leqslant mT_1, \ldots, T_n.P_n) \in \mathcal{L}(p)$ and $\sharp T_1T_2 \ldots T_n^{\mathcal{T}}(p, P_n) > m$. According to the definition of $T_1T_2 \ldots T_n^{\mathcal{T}}(p, P_n)$, let $value(t_{ji})$ be the value of node t_{ji}, this implies that there exist $\langle t_{11}, \ldots, t_{1n}\rangle, \ldots, \langle t_{m+1,1}, \ldots, t_{m+1,n}\rangle$ such that $\langle p, value(t_{ji})\rangle \in \mathcal{E}_D(T_i)$, and $\{p, \langle T_1, \ldots, T_n\rangle, \langle t_{j1}, \ldots, t_{jn}\rangle, P_n\} \in DC^{\mathtt{F}}$, for all $1 \leq i \leq n$ and $1 \leq j < k \leq m+1$. Therefore the $\leqslant$-rule is still applicable, which is a contradiction to the completeness of $\mathtt{F}$. Thus the assumption $\sharp T_1T_2 \ldots T_n^{\mathcal{T}}(p, P_n) > m$ is false. So we have $\sharp T_1T_2 \ldots T_n^{\mathcal{T}}(p, P_n) \leqslant m$.
- (P16): Assume $\{\leqslant mT_1, \cdots, T_n.P_n, \geqslant mT_1, \cdots, T_n.P_n\} \cap \mathcal{L}(p) \neq \emptyset$, $\langle p, t_i\rangle \in \mathcal{E}_{\mathbf{D}}(T_i), 1 \leq i \leq n$, thus $\langle t_1, \ldots, t_n\rangle$ is a $T_1T_2 \ldots T_n$-successors of $\mathtt{Tail}(p)$. Let $value(t_i)$ be the value of t_i: (1) if $\{p, \langle T_1, \ldots, T_n\rangle, \langle t_1, \ldots, t_n\rangle, P_n\} \in DC^{\mathtt{F}}$, we have $DC^{\mathcal{T}}(p, T_1, \ldots, T_n, value(t_1), \ldots, value(t_n), P_n) = true$; (2) if $\{p, \langle T_1, \ldots, T_n\rangle, \langle t_1, \ldots, t_n\rangle, \neg P_n\} \in DC^{\mathtt{F}}$, we have $DC^{\mathcal{T}}(p, T_1, \ldots, T_n, value(t_1), \ldots, value(t_n), \neg P_n) = true$.

■

Lemma 8 (Completeness) If a $\mathcal{SHOQ}(\mathbf{D_n})$-concept D in NNF has a tableau w.r.t. $\mathcal{R}$, then the expansion rules can be applied to D and $\mathcal{R}$ such that they yield a complete, clash-free completion forest.

Proof: Let $\mathcal{T} = (\mathbf{S}, \mathcal{L}, \mathcal{E}_A, \mathcal{E}_{\mathbf{D}})$ be a tableau for D w.r.t. a role box $\mathcal{R}$. We use $\mathcal{T}$ to guide the application of the non-deterministic rules. We define a function π, mapping the nodes of the forest $\mathtt{F}$ to $\mathbf{S} \cup \Delta_{\mathbf{D}}$ such that $\mathcal{L}(x) \subseteq \mathcal{L}(\pi(x))$; $\langle \pi(x), \pi(y)\rangle \in \mathcal{E}_A$ if: 1. $\pi(y) \in \mathbf{S}$ and y is an R_A-successor of x, or 2. $\uparrow (R, \{o\}) \in \mathcal{L}(x)$ and $y = x_{\{o\}}$; $\langle \pi(x), \pi(y)\rangle \in \mathcal{E}_{\mathbf{D}}$ if $\pi(y) \notin \mathbf{S}$ and y is an $R_{\mathbf{D}}$-successor of x; $x \neq y$ implies $\pi(x) \neq \pi(y)$; $\langle y_{j1}, \ldots, y_{jn}\rangle \dot{\neq} \langle y_{k1}, \ldots, y_{kn}\rangle$ implies $\langle \pi(y_{j1}), \ldots, \pi(y_{jn})\rangle \dot{\neq} \langle \pi(y_{k1}), \ldots, \pi(y_{kn})\rangle$ for $y_{j1}, \ldots, y_{jn}, y_{k1}, \ldots, y_{kn} \notin \mathbf{S}$. $(*)$

We only have to consider the various rules about number restriction on concrete roles.

- The $\geqslant_P$-rule: If $\geqslant mT_1, \ldots, T_n.P_n \in \mathcal{L}(x)$, then $\geqslant mT_1, \ldots, T_n.P_n \in \mathcal{L}(\pi(x))$. Since $\mathcal{T}$ is a tableau, (P14) of Definition 3 implies that $\sharp T_1T_2 \ldots T_n^{\mathcal{T}}(\pi(x), P_n) \geqslant m$. Hence there are m tuples $\langle t_{11}, \ldots, t_{1n}\rangle, \ldots, \langle t_{m1}, \ldots, t_{mn}\rangle$, such that $\langle \pi(x), t_{ji}\rangle \in \mathcal{E}_{\mathbf{D}}$, $\langle t_{j1}, \ldots, t_{jn}\rangle \dot{\neq} \langle t_{k1}, \ldots, t_{kn}\rangle$, and $DC^{\mathcal{T}}(\pi(x), T_1, \ldots, T_n, t_{j1}, \ldots, t_{jn}, P_n) = true$, for $1 \leq i \leq n$ and $1 \leq j < k \leq m$. The $\geqslant_P$-rule generates m new $T_1T_2 \ldots T_n$-successors $\langle y_{11}, \ldots, y_{1n}\rangle, \ldots, \langle y_{m1}, \ldots, y_{mn}\rangle$. By setting $\pi' := \pi[y_{ji} \mapsto t_{ji}](1 \leq i \leq n, 1 \leq j < k \leq m)$, one obtains a function π' that satisfies $(*)$ for the modified forest.
- The $\leqslant_P$-rule: If $\leqslant mT_1, \ldots, T_n.P_n \in \mathcal{L}(x)$, then $\leqslant mT_1, \ldots, T_n.P_n \in \mathcal{L}(\pi(x))$. Since $\mathcal{T}$ is a tableau, (P15) of Definition 3 implies $\sharp T_1T_2 \ldots T_n^{\mathcal{T}}(\pi(x), P_n) \leqslant m$. If the $\leqslant_P$-rule is applicable, we have $\sharp DCSuccessors^{\mathtt{F}}(x, T_1, \ldots, T_n, P_n) > m$, which implies that there are at least $m+1$ $T_1T_2 \ldots T_n$-successors $\langle y_{11}, \ldots, y_{1n}\rangle, \ldots, \langle y_{m+1,1}, \ldots, y_{m+1,n}\rangle$ such that $\{x, \langle T_1, \ldots, T_n\rangle, \langle y_{j1}, \ldots, y_{jn}\rangle, P_n\} \in DC^{\mathtt{F}}$, for $1 \leq j \leq m+1$. Thus, there must be two $\langle y_{j1}, \ldots, y_{jn}\rangle$ and $\langle y_{k1}, \ldots, y_{kn}\rangle$ among the $m+1$ $T_1T_2 \ldots T_n$-successors such that $\langle \pi(y_{j1}), \ldots, \pi(y_{jn})\rangle = \langle \pi(y_{k1}), \ldots, \pi(y_{kn})\rangle$ (otherwise $\sharp T_1T_2 \ldots T_n^{\mathcal{T}}(\pi(x), P_n) > m$ would hold). This implies $\langle y_{j1}, \ldots, y_{jn}\rangle \dot{\neq} \langle y_{k1}, \ldots, y_{kn}\rangle$ cannot hold because of $(*)$. Hence the $\leqslant_P$-rule can be applied without violating $(*)$.
- The $choose_P$-rule: If $\{\leqslant mT_1, \cdots, T_n.P_n, \geqslant mT_1, \cdots, T_n.P_n\} \cap \mathcal{L}(x) \neq \emptyset$, we have $\{\leqslant mT_1, \cdots, T_n.P_n, \geqslant mT_1, \cdots, T_n.P_n\} \cap \mathcal{L}(\pi(x)) \neq \emptyset$, and if there are $T_1T_2 \ldots T_n$-successors $\langle y_1, \ldots, y_n\rangle$ of x, then $\langle \pi(x), \pi(y_i)\rangle \in \mathcal{E}_{\mathbf{D}}, 1 \leq i \leq n$, due to $(*)$. Since $\mathcal{T}$ is a tableau, (P16) of Definition 3 implies either $DC^{\mathcal{T}}(\pi(x), T_1, \ldots, T_n, \pi(y_1), \ldots, \pi(y_n), P_n) = true$, or $DC^{\mathcal{T}}(\pi(x), T_1, \ldots, T_n, \pi(y_1), \ldots, \pi(y_n), \neg P_n) = true$. Hence the $choose_P$-rule can accordingly either set $DC^{\mathtt{F}} \longrightarrow DC^{\mathtt{F}} \cup \{x, \langle T_1, \ldots, T_n\rangle, \langle y_1, \ldots, y_n\rangle, P_n\}$ or set $DC^{\mathtt{F}} \longrightarrow DC^{\mathtt{F}} \cup \{x, \langle T_1, \ldots, T_n\rangle, \langle y_1, \ldots, y_n\rangle, \neg P_n\}$.

Whenever a rule is applicable to $\mathtt{F}$, it can be applied in a way that maintains $(*)$, and, from Lemma 6, we have that any sequence of rule applications must terminate. Since $(*)$ holds, any forest generated by these rule-applications must be clash-free. This can be seen from the condition described in [7] plus the following:

- If $\mathtt{F}$ does not satisfy $DC^{\mathtt{F}}$, there must be some concrete nodes from which no values mapping satisfies all the relevant predicates, and therefore there can be no values satisfying all of properties (P12) to (P16).

- F cannot contain a node x with $\leqslant m T_1, \ldots, T_n.P_n \in \mathcal{L}(x)$, and $m+1$ $T_1 T_2 \ldots T_n$-successors $\langle t_{11}, \ldots, t_{1n}\rangle, \ldots, \langle t_{m+1,1}, \ldots, t_{m+1,n}\rangle$ of x with $P_n(i) \in \mathcal{L}(t_{ji})$, $\langle t_{j1}, \ldots, t_{jn}\rangle \dot{\neq} \langle t_{k1}, \ldots, t_{kn}\rangle$ and $DC^{\mathtt{F}}(x, t_{j1}, \ldots, t_{jn}, P_n) = true$, for all $1 \leq i \leq n, 1 \leq j < k \leq m{+}1$, and, since $\langle t_{j1}, \ldots, t_{jn}\rangle \dot{\neq} \langle t_{k1}, \ldots, t_{kn}\rangle$ implies $\langle \pi(t_{j1}), \ldots, \pi(t_{jn})\rangle \dot{\neq} \langle \pi(t_{k1}), \ldots, \pi(t_{kn})\rangle$, $\sharp T_1 T_2 \ldots T_n^{\mathcal{T}}(\pi(x), P_n) > n$ would hold which contradicts (P15) of Definition 3. ∎

As an immediate consequence of Lemmas 2,4,5 and 6, the completion algorithm always terminates, and answers with "D is satisfiable w.r.t. $\mathcal{R}$" iff. D has a tableau T. Next, subsumption can be reduced to (un)satisfiability. Finally, $\mathcal{SHOQ}(\mathbf{D_n})$ can internalise general concept inclusion axions [5]. However, in the presence of nominals, we must also add $\exists O.o_1 \sqcap \cdots \sqcap \exists O.o_l$ to the concept internalising the general concept inclusion axioms to make sure that the universal role O indeed reaches all nominals O_i occuring in the input concept and terminology. Thus, we can decide these inference problems also w.r.t. terminologies.

Theorem 9 The tableau algorithm presented in Definition 5 is a decision procedure for satisfiability and subsumption of $\mathcal{SHOQ}(\mathbf{D_n})$-concepts w.r.t. terminologies.

5 Discussion

As we have seen, unary datatype predicates are usually not enough, while n-ary datatype predicates are often necessary in modelling the "concrete properties" of real world entities. Furthermore, datatype number restrictions are very expressive that e.g., with them, we can define single/multiple-value datatype attributes. Therefore, we have extended $\mathcal{SHOQ}(\mathbf{D})$ with n-ary datatype predicates and datatype number restrictions to give the $\mathcal{SHOQ}(\mathbf{D_n})$ DL. We have shown that the decision procedure for concept satisfiability and subsumption is still decidable in $\mathcal{SHOQ}(\mathbf{D_n})$. An implementation based on the FaCT system is planned, and will be used to test empirical performance.

With its support for both nominals and n-ary datatype predicates with datatype number restrictions, $\mathcal{SHOQ}(\mathbf{D_n})$ is well suited to provide reasoning support for ontology languages in general, and Semantic Web ontology languages in particular. As future work, it would be interesting to study the datatype number restrictions in the Semantic Web applications. It is also important to extend current optimisation techniques to cope with nominals used in the logic. The $\mathcal{SHOQ}(\mathbf{D_n})$ DL decision procedure is similar to those of the $\mathcal{SHIQ}$ DL implemented in the successful FaCT system, and should be amenable to a similar range of performance enhancing optimisations. Thirdly, ABox reasoning and query answering in $\mathcal{SHOQ}(\mathbf{D_n})$ are also very interesting, since these efforts will make more reasoning services available, e.g., querying services, to the Web ontology languages, such as DAML+OIL.

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References

[1] F. Baader and Philipp Hanschke. A Scheme for Integrating Concrete Domains into Concept Languages. In *IJCAI-91*, pages 452–457, 1991.

[2] Tim Berners-lee. Semantic Web Road Map. W3C Design Issues. URL`http://www.w3.org/DesignIssues/Semantic.html`, Oct. 1998.

[3] J. Broekstra, M. Klein, S. Decker, D. Fensel, F. can Harmelen, and I. Horrocks. Enabling knowledge representation on the Web by extending RDF Schema, Nov. 2000.

[4] Volker Haarslev, Ralf Mller, and Michael Wessel. The Description Logic ALCNHR+ Extended with Concrete Domains: A Practically Motivated Approach. In *Proceedings of International Joint Conference on Automated Reasoning, IJCAR'2001, R. Gor, A. Leitsch, T. Nipkow (Eds.),Siena, Italy, Springer-Verlag, Berlin.*, pages 29–44, Jun. 2001.

[5] I. Horrocks, U. Sattler, and S. Tobies. Practical Reasoning for Expressive Description Logics. In H. Ganzinger, D. McAllester, and A. Voronkov, editors, *Proceedings of the 6th International Conference on Logic for Programming and Automated Reasoning (LPAR'99)*, number 1705 in Lecture Notes in Artificial Intelligence, pages 161–180. Springer-Verlag, 1999.

[6] Ian Horrocks and Peter F. Patel-Schneider. The Generation of DAML+OIL. Aug. 2001. The 2001 International Workshop on Description Logics.

[7] Ian Horrocks and U. Sattler. Ontology Reasoning for the Semantic Web. In *In B. Nebel, editor, Proc. of the 17th Int. Joint Conf. on Artificial Intelligence (IJCAI'01), Morgan Kaufmann*, pages 199–204, 2001.

[8] C. Lutz. NExpTime-complete Description Logics with Concrete Domains . LuFG Theoretical Computer Science, RWTH Aachen, Germany, 2000.

[9] Jeff Z. Pan. Web Ontology Reasoning in the SHOQ(Dn) Description Logic. In *Proceedings of the Methods for Modalities 2 (M4M-2)*, Nov 2001. ILLC, University of Amsterdam, URL `http://www.cs.man.ac.uk/~panz/Zhilin/download/Paper/Pan-shoqdn-2001.pdf`.

[10] Jeff Z. Pan and Ian Horrocks. Metamodeling Architecture of Web Ontology Languages. In *Proceeding of the Semantic Web Working Symposium (SWWS)*, July 2001. URL `http://www.cs.man.ac.uk/~panz/Zhilin/download/Paper/Pan-Horrocks-rdfsfa-2001.pdf`.