Delta lenses as coalgebras for a comonad

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Abstract

Delta lenses are a kind of morphism between categories which are used to model bidirectional transformations between systems. Classical state-based lenses, also known as very well-behaved lenses, are both algebras for a monad and coalgebras for a comonad. Delta lenses generalise state-based lenses, and while delta lenses have been characterised as certain algebras for a semi-monad, it is natural to ask if they also arise as coalgebras.

This short paper establishes that delta lenses are coalgebras for a comonad, through showing that the forgetful functor from the category of delta lenses over a base, to the category of cofunctors over a base, is comonadic. The proof utilises a diagrammatic approach to delta lenses, and clarifies several results in the literature concerning the relationship between delta lenses and cofunctors. Interestingly, while this work does not generalise the corresponding result for state-based lenses, it does provide new avenues for exploring lenses as coalgebras.

Keywords

delta lens, cofunctor, coalgebra, bidirectional transformation

1. Introduction

The goal of understanding various kinds of lenses as mathematical structures has been an ongoing program in the study of bidirectional transformations. For example, *very well-behaved lenses* [1], also known as *state-based lenses* [2], have been understood as both algebras for a monad [3] and coalgebras for a comonad [4, 5]. A generalisation of state-based lenses called *category lenses* [6] were also introduced as algebras for a monad, based on classical work in 2-category theory on split opfibrations [7]. Another kind of lens between categories called a *delta lens* [8] was shown to be a certain algebra for a semi-monad [9], however it remained open as to whether delta lenses could also be characterised as (co)algebras for a (co)monad.

The purpose of this short paper is to characterise delta lenses as coalgebras for a comonad (Theorem 9). The proof of this simple result builds upon and clarifies several recent advances in the theory of delta lenses.

In 2017, Ahman and Uustalu introduced *update-update lenses* [2] as morphisms of *directed containers* [10], which are equivalent to certain morphisms called *cofunctors* between categories [11]. In the same paper, they show explicitly how, using the notation of directed containers, delta lenses may be understood as cofunctors with additional structure.

In earlier work [12] from 2016, Ahman and Uustalu also provide a construction on morphisms of directed containers which yields a *split pre-opcleavage* for a functor; in other words, they

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show how cofunctors may be turned into delta lenses. We show that this construction is actually a right adjoint to the forgetful functor from delta lenses to cofunctors (Lemma 8), and that the coalgebras for the comonad generated from this adjunction are delta lenses (Theorem 9).

In 2020, a diagrammatic characterisation of delta lenses was introduced by the current author [13], building upon an earlier characterisation of cofunctors as spans [14]. This diagrammatic approach is utilised throughout this paper, and leads to another simple characterisation of delta lenses (Proposition 6).

Overview of the paper and related work

This section provides an informal overview of the paper, together with further commentary on the background, and references to related work. The goal is to provide a conceptual understanding of the results; later sections will be dedicated to the formal mathematics.

Section 2 contains the mathematical background required for the main results, which are presented in Section 3. Consequences of the main result and concluding remarks are in Section 4.

Throughout the paper we make the assumption that a *system*, whatever that may be, can be understood as a category. The objects of this category are the *states* of the system, while the morphisms are the *transitions* (or *deltas*) between system states.

Delta lenses were introduced in [8, Definition 4] to model bidirectional transformations between systems when they are understood as categories. The GET of a delta lens is a functor $f\colon A \to B$ from the *source category* A to the *view category* B, while the PUT is a certain kind of function (that this paper calls a *lifting operation*) satisfying axioms analogous to the classical lens laws. A slightly modified definition of delta lens appeared in [9, Definition 1], however this definition still seemed to be ad hoc, and made it difficult to prove deep results without checking many details.

The definition of delta lens (Definition 4) given in this paper is based on a diagrammatic characterisation which first appeared in [13, Corollary 20], by representing the PUT in terms of bijective-on-objects functors (Definition 1) and discrete opfibrations (Definition 2). This diagrammatic approach provides a natural framework for studying delta lenses using category theory, and has the benefit of allowing for very simple (albeit more abstract) proofs. This approach will be utilised throughout this paper, although in many places we will also include explicit descriptions of constructions using the traditional definition of a delta lens.

A key idea presented in [2, 13] is that the GET and PUT of a delta lens can be separated into functors and *cofunctors* (Definition 3), respectively. Intuitively, a cofunctor can be understood as a delta lens without any information on how the GET acts on morphisms; it is the minimum amount of structure needed to specify a PUT operation between categories. It was shown in the paper [2] that delta lenses are cofunctors with additional structure. In this paper, we aim to show that said structure arises coalgebraically via a comonad.

Both delta lenses and cofunctors are predominantly understood and studied as *morphisms* between categories, however to prove that delta lenses are cofunctors equipped with coalgebraic structure, it is necessary for them to be understood as *objects*. Therefore this paper introduces a new category $\mathsf{Cof}(B)$, whose objects are cofunctors into a fixed category B (Definition 5). The category $\mathsf{Lens}(B)$, whose objects are delta lenses into a fixed category B, was previously studied in [15, 16]. Surprisingly, we show that the category $\mathsf{Lens}(B)$ can be defined (Definition 7) as the slice category $\mathsf{Cof}(B)/1_B$. Not only does this provide a new characterisation of delta lenses in term of cofunctors (Proposition 6), but also provides the insight that the canonical forgetful functor $L\colon \mathsf{Lens}(B) \to \mathsf{Cof}(B)$, which takes a delta lens to its underlying PUT cofunctor, is a projection from a slice category.

Finally, proving that delta lenses are coalgebras for a comonad on $\mathsf{Cof}(B)$ amounts to showing that the forgetful functor $L\colon \mathsf{Lens}(B) \to \mathsf{Cof}(B)$ is *comonadic* (Theorem 9). A necessary condition is that L has a right adjoint R (Lemma 8), which constructs the *cofree delta lens* from each cofunctor in $\mathsf{Cof}(B)$. This construction first appeared explicitly in [12, Section 3.2], however it was not obviously a right adjoint — or even a functor — and it was disconnected from the context of cofunctors and delta lenses. Both Lemma 8 and Theorem 9 admit straightforward proofs, with the benefit of the diagrammatic approach to cofunctors and delta lenses.

Notation and conventions

This section outlines some of the notation and conventions used in the paper. Given a category A, its underlying set (or discrete category) of objects is denoted A_0. Given a functor $f\colon A \to B$, its underlying object assignment is denoted $f_0\colon A_0 \to B_0$. Similarly, a cofunctor $\varphi\colon A \nrightarrow B$ will have an underlying object assignment $\varphi_0\colon A_0 \to B_0$. Thus the orientation of a cofunctor agrees with the orientation of its underlying object assignment (this convention is chosen to agree with the orientation of delta lenses, however this choice is not uniform in the literature on cofunctors). The operation cod sends each morphism to its *codomain* or *target* object.

2. Prerequisites for the main result

We first recall two special classes of functors, which we will use as the building blocks for defining cofunctors and delta lenses. New contributions in this section include the category $\mathsf{Cof}(B)$ whose objects are cofunctors (Definition 5), and the characterisation of delta lenses as certain morphisms therein (Proposition 6).

Definition 1. A functor $f\colon A \to B$ is *bijective-on-objects* if its underlying object assignment $f_0\colon A_0 \to B_0$ is a bijection.

Definition 2. A functor $f\colon A \to B$ is a *discrete opfibration* if for all pairs,

$$(a \in A,\ u\colon fa \to b \in B)$$

there exists a unique morphism $w\colon a \to a'$ in A such that $fw = u$.

Definition 3. A *cofunctor* $\varphi\colon A \nrightarrow B$ between categories is a span of functors,

$$\begin{array}{ccc} & X & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\overline{\varphi}} \\ A & & B \end{array} \tag{1}$$

where φ is a bijective-on-objects functor and $\overline{\varphi}$ is a discrete opfibration.

Alternatively, a cofunctor $\varphi\colon A \nrightarrow B$ consists of a function $\varphi_0\colon A_0 \to B_0$, together with a *lifting operation* φ, which assigns each pair $(a \in A,\ u\colon \varphi_0 a \to b \in B)$ to a morphism $\varphi(a, u)\colon a \to a'$ in A, such that the following axioms are satisfied:

(1) $\varphi_0 \operatorname{cod}\big(\varphi(a, u)\big) = \operatorname{cod}(u)$;
(2) $\varphi(a, 1_{\varphi_0 a}) = 1_a$;
(3) $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a, u)$, where $a' = \operatorname{cod}\big(\varphi(a, u)\big)$.

Definition 4. A *delta lens* $(f, \varphi)\colon A \rightleftharpoons B$ between categories is a commutative diagram of functors,

$$\begin{array}{ccc} & X & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\overline{\varphi}} \\ A & \xrightarrow[f]{} & B \end{array} \tag{2}$$

where φ is a bijective-on-objects functor and $\overline{\varphi}$ is a discrete opfibration.

We can also describe a delta lens $(f, \varphi)\colon A \rightleftharpoons B$ as consisting of a functor $f\colon A \to B$ together with a *lifting operation* φ, which assigns each pair $(a \in A,\ u\colon fa \to b \in B)$ to a morphism $\varphi(a, u)\colon a \to a'$ in A, such that the following axioms are satisfied:

(1) $f\varphi(a, u) = u$;
(2) $\varphi(a, 1_{fa}) = 1_a$;
(3) $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a, u)$, where $a' = \operatorname{cod}\big(\varphi(a, u)\big)$.

Every delta lens $(f, \varphi)\colon A \rightleftharpoons B$ has an underlying functor $f\colon A \to B$ and an underlying cofunctor $\varphi\colon A \nrightarrow B$, and their corresponding underlying object assignments are equal; that is, $f_0 = \varphi_0$.

Definition 5. For each category B, there is a category $\mathsf{Cof}(B)$ of *cofunctors over the base* B whose objects are cofunctors with codomain B, and whose morphisms are given by commutative diagrams of functors of the form:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\gamma} \\ X & \xrightarrow{\overline{h}} & Y \\ & {\scriptstyle\overline{\varphi}}\searrow \quad \swarrow{\scriptstyle\overline{\gamma}} & \\ & B & \end{array} \tag{3}$$

Equivalently, a morphism in $\mathsf{Cof}(B)$ from a cofunctor $\varphi\colon A \nrightarrow B$ to a cofunctor $\gamma\colon C \nrightarrow B$ consists of a functor $h\colon A \to C$ such that $\gamma_0 ha = \varphi_0 a$ for all $a \in A$, and $h\varphi(a, u) = \gamma(ha, u)$ for all pairs $(a \in A,\ u\colon \varphi_0 a \to b \in B)$. The functor $\overline{h}\colon X \to Y$ is then uniquely induced from this data. Intuitively, if A and C are understood as *source categories* with a fixed *view category* B, then the morphisms in $\mathsf{Cof}(B)$ are functors between the source categories which preserve the chosen lifts, given by the corresponding cofunctors, from the view category.

Proposition 6. *Every delta lens $(f, \varphi)\colon A \rightleftharpoons B$ is equivalent to a morphism in* $\mathsf{Cof}(B)$ *whose codomain is the trivial cofunctor on B.*

Proof. Consider the morphism in $\mathsf{Cof}(B)$ given by the commutative diagram of functors:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle 1_B} \\ X & \xrightarrow{\ \overline{\varphi}\ } & B \\ & {\scriptstyle\overline{\varphi}}\searrow \qquad \swarrow{\scriptstyle 1_B} & \\ & B & \end{array} \tag{4}$$

The upper commutative square describes a delta lens as given in Definition 4. Conversely, every delta lens may be depicted as a morphism in $\mathsf{Cof}(B)$ in this way. □

We can unpack (4) using the explicit characterisation of morphisms in $\mathsf{Cof}(B)$ to obtain the precise difference between cofunctors and delta lenses, in terms of objects and morphisms. Namely, the diagram (4) states that a delta lens corresponds to a cofunctor $\varphi\colon A \nrightarrow B$ together with a functor $f\colon A \to B$ such that $fa = \varphi_0 a$ for all $a \in A$, and $f\varphi(a, u) = u$ for all pairs $(a \in A,\ u\colon fa \to b \in B)$.

Definition 7. For each category B, we define the category of *delta lenses over the base B* to be the slice category $\mathsf{Lens}(B) := \mathsf{Cof}(B) \,/\, 1_B$, where 1_B is the trivial cofunctor on B.

By Proposition 6, the objects of $\mathsf{Lens}(B)$ are delta lenses with codomain B, represented as a morphism into the trivial cofunctor as shown in (4). The morphisms in $\mathsf{Lens}(B)$ are given by morphisms (3) in $\mathsf{Cof}(B)$ such that the following pasting condition holds:

$$\begin{array}{ccccc} A & \xrightarrow{\ h\ } & C & \xrightarrow{\ g\ } & B \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle 1_B} \\ X & \xrightarrow{\ \overline{h}\ } & Y & \xrightarrow{\ \overline{\gamma}\ } & B \\ & {\scriptstyle\overline{\varphi}}\searrow & \downarrow{\scriptstyle\overline{\gamma}} & \swarrow{\scriptstyle 1_B} & \\ & & B & & \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle 1_B} \\ X & \xrightarrow{\ \overline{\varphi}\ } & B \\ & {\scriptstyle\overline{\varphi}}\searrow \qquad \swarrow{\scriptstyle 1_B} & \\ & B & \end{array} \tag{5}$$

In other words, the only additional requirement on a morphism $h\colon A \to C$ between delta lenses over B, compared to a morphism between cofunctors over B, is that $g \circ h = f$. This is opposed to just requiring $\gamma_0 ha = \varphi_0 a$ on objects (where recall for delta lenses, the underlying object assignments for the functor and cofunctor are equal, that is, $g_0 = \gamma_0$ and $f_0 = \varphi_0$).

There is a canonical forgetful functor,

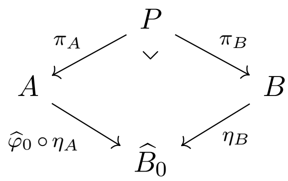
$$L\colon \mathsf{Lens}(B) \longrightarrow \mathsf{Cof}(B)$$

which assigns every delta lens to its underlying cofunctor. This forgetful functor is the focus of the main result in the following section.

3. Main result

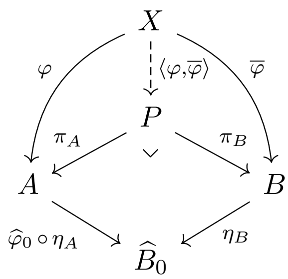
While not every cofunctor may be given the structure of a delta lens, Ahman and Uustalu [12] developed a method which constructs a delta lens from any cofunctor. To understand their construction, first recall that the underlying objects functor $(-)_0 \colon \mathsf{Cat} \to \mathsf{Set}$ has a right adjoint $\widehat{(-)} \colon \mathsf{Set} \to \mathsf{Cat}$ which takes each set X to the *codiscrete category* $\widehat{X}$.

Given a cofunctor $\varphi \colon A \nrightarrow B$ with underlying object assignment $\varphi_0 \colon A_0 \to B_0$, we may construct the following pullback in Cat:



(6)

Here $\eta_B \colon B \to \widehat{B}_0$ is the component of the unit for the adjunction at B, and $\widehat{\varphi}_0 \circ \eta_A$ the component of the unit at A followed by image of φ_0 under the right adjoint. Using the universal property of the pullback, we have the following:



(7)

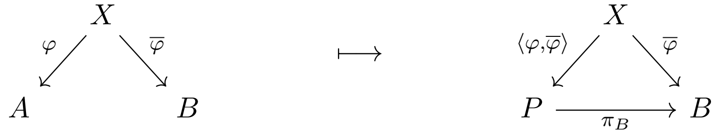
Since η_B is bijective-on-objects, the projection functor π_A is also bijective-on-objects which, together with the functor φ, implies that $\langle \varphi, \overline{\varphi} \rangle \colon X \to P$ is bijective-on-objects, due to the properties of bijections at the level of objects. Thus, the upper right triangle in (7) defines a delta lens $P \rightleftharpoons B$.

The category P has the same objects as A, but morphisms $a \to a'$ in P are given by pairs of the form $(w \colon a \to a' \in A, u \colon \varphi_0 a \to \varphi_0 a' \in B)$. The functor $\pi_B \colon P \to B$ projects to the second arrow in this pair. The lifting operation which makes this functor into a delta lens is induced by the lifting operation of the original cofunctor; it takes an object $a \in P$ and a morphism $u \colon \varphi_0 a \to b \in B$ to the morphism $\big(\varphi(a, u) \colon a \to a', u \colon \varphi_0 a \to b\big)$ in P.

We now show that this construction due to Ahman and Uustalu is universal, in the sense that it provides a right adjoint to the functor taking a delta lens to its underlying cofunctor.

Lemma 8. *The forgetful functor $L \colon \mathsf{Lens}(B) \to \mathsf{Cof}(B)$ has a right adjoint.*

Proof. Using the construction in (7), define the functor $R \colon \mathsf{Cof}(B) \to \mathsf{Lens}(B)$ by the assignment:



(8)

We describe the components of the unit and counit for the adjunction $L \dashv R$ and omit the detailed checks that the triangle identities hold.

Given a cofunctor $\varphi\colon A \nrightarrow B$ the component of the counit is given by:

$$\begin{array}{ccc} P & \xrightarrow{\pi_A} & A \\ {\scriptstyle\langle\varphi,\overline{\varphi}\rangle}\uparrow & & \uparrow{\scriptstyle\varphi} \\ X & = & X \\ & {\scriptstyle\overline{\varphi}}\searrow \quad \swarrow{\scriptstyle\overline{\varphi}} & \\ & B & \end{array} \tag{9}$$

Given a delta lens $(f, \varphi)\colon A \rightleftharpoons B$ the component of the unit is given by:

$$\begin{array}{ccccc} A & \xrightarrow{\langle 1_A, f\rangle} & P & \xrightarrow{\pi_B} & B \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\langle\varphi,\overline{\varphi}\rangle} & & \uparrow{\scriptstyle 1_B} \\ X & = & X & \xrightarrow{\overline{\varphi}} & B \\ & {\scriptstyle\overline{\varphi}}\searrow & \downarrow{\scriptstyle\overline{\varphi}} & \swarrow{\scriptstyle 1_B} & \\ & & B & & \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle 1_B} \\ X & \xrightarrow{\overline{\varphi}} & B \\ & {\scriptstyle\overline{\varphi}}\searrow \quad \swarrow{\scriptstyle 1_B} & \\ & B & \end{array} \tag{10}$$

The above diagrams show that the pasting condition required in (5) is satisfied. □

Theorem 9. *The forgetful functor $L\colon \mathsf{Lens}(B) \to \mathsf{Cof}(B)$ is comonadic.*

Proof. By Lemma 8, the functor L has a right adjoint R. To prove that L is comonadic, it remains to show that the category of coalgebras for the induced comonad LR on $\mathsf{Cof}(B)$ is equivalent to $\mathsf{Lens}(B)$.

Given a cofunctor $\varphi\colon A \nrightarrow B$, a coalgebra structure map is given by a morphism in $\mathsf{Cof}(B)$ of the form:

$$\begin{array}{ccc} A & \xrightarrow{h} & P \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\langle\varphi,\overline{\varphi}\rangle} \\ X & \xrightarrow{\overline{h}} & X \\ & {\scriptstyle\overline{\varphi}}\searrow \quad \swarrow{\scriptstyle\overline{\varphi}} & \\ & B & \end{array} \tag{11}$$

However compatibility with the counit forces $\overline{h} = 1_X$ and $h = \langle 1_A, f\rangle$, where $f\colon A \to B$ is a functor such that $f \circ \varphi = \overline{\varphi}$. Compatibility with the comultiplication doesn't add any further conditions. Therefore, a coalgebra for the comonad LR on $\mathsf{Cof}(B)$ is equivalent to a delta lens $(f, \varphi)\colon A \rightleftharpoons B$. □

This theorem establishes the result stated in the title of the paper, that delta lenses (2) are coalgebras (11) for a comonad.

4. Concluding remarks

In this paper, the category $\mathsf{Lens}(B)$ of delta lenses over the base B was characterised as the category of coalgebras for a comonad on the category $\mathsf{Cof}(B)$ of cofunctors over the base B. This brings together recent results in the study of delta lenses and cofunctors. In particular, we have shown that the extra structure on cofunctors given in Ahman and Uustalu's [2] characterisation of delta lenses is coalgebraic, and that their construction of a delta lens from cofunctor in the paper [12] is precisely the cofree delta lens on a cofunctor. Throughout we have also shown how the abstract diagrammatic approach to delta lenses, first introduced in [13], has led to concise proofs of these results, and offers a clear perspective on the relationship between these ideas.

Aside from clarification and development of theory, the results presented in this paper have several other mathematical consequences. For example, the functor $L\colon \mathsf{Lens}(B) \to \mathsf{Cof}(B)$ creates all colimits which exist in $\mathsf{Cof}(B)$. Thus we can take the coproduct of a pair of cofunctors in $\mathsf{Cof}(B)$, and automatically know how to construct the coproduct of the corresponding delta lenses in $\mathsf{Lens}(B)$.

Another consequence from the unit (10) of the adjunction between $\mathsf{Cof}(B)$ and $\mathsf{Lens}(B)$ is that every delta lens factorises into a bijective-on-objects functor followed by a cofree lens. Intuitively, this allows us to first pair every transition in the source category A with a transition in the view category B via the functor part $f\colon A \to B$ of the delta lens,

$$w\colon a \to a' \in A \qquad \longmapsto \qquad (w\colon a \to a' \in A, fw\colon fa \to fa' \in B)$$

then consider the update propagation determined by the cofunctor part $\varphi\colon A \nrightarrow B$ of the delta lens. The cofree delta lens on a cofunctor behaves much like an analogue of *constant complement* state-based lenses, except that the complement is with respect to morphisms rather than objects.

While the main contributions of this paper are mathematical, it is hoped that these results also prompt new ways of understanding delta lenses. For example, previously state-based lenses have been considered from a "Put-based" perspective [17, 18], however this approach could also be adapted to the setting of delta lenses. Rather than starting with a Get functor between systems and then asking how we might construct a delta lens, we might instead start with a Put cofunctor and then ask for ways in which this can be given the structure of a delta lens. This shift of focus is subtle but important, especially in the context of the ideas in [2], as it is arguably the Put structure (rather than the Get structure) which is central to the study of bidirectional transformations and lenses.

On an separate note, it is worth remarking on the similarity between the main result of this paper and the classical result stating that very well-behaved lenses are coalgebras for a comonad [4, 5]. Despite the clear analogy between them, and the inspiration that this paper derives from the classical result, it seems that they are unrelated at a mathematical level. The classical result relies on Set being a cartesian closed category, and arises from the adjunction $(-) \times B \dashv [B, -]$, whereas the results in this paper arise from a different adjunction, and don't require any aspect of cartesian closure.

There are many questions to be explored in future work. For instance, it is natural to ask if $\mathsf{Lens}(B)$ is comonadic over other categories (such as Cat as was suggested by an anonymous reviewer), or if split opfibrations (also known as c-lenses [6]) are also comonadic over $\mathsf{Cof}(B)$. In recent work by the current author, it has been demonstrated that delta lenses arise as algebras for a monad on Cat/B, providing a dual to the main result of this paper and strengthening the previous work of Johnson and Rosebrugh [9]. Finally, given the importance of the category $\mathsf{Lens}(B)$ in the study of *symmetric lenses* [15, 16], it is also hoped that the coalgebraic perspective provides new insights into this area, and this will be the subject of further investigation.

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