

ANALYSIS OF THE PARACONSISTENCY IN SOME LOGICS

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Abstract. In Artificial Intelligence, as well as in data base updating or in the design of intelligent agents, it is necessary the use of contradictory information. For that, it is useful to direct our attention to paraconsistent logics. The goal of the present survey is to offer an initial analysis on paraconsistency in some logics like Pac, RM₃, Lukasiewicz, G₃.

1 INTRODUCTION TO PARACONSISTENCY

This section is devoted to introduce the concepts and basic principles of paraconsistency, as well as its properties.

Definition 1. [2] *Given a set, For, of formulas, we say that \vdash defines a consequence relation on For if it is such that $\vdash \subseteq P(\text{For}) \times \text{For}$, and if the following conditions hold:*

- Con1.** $A \in \Gamma \implies \Gamma \vdash A$ *Reflexivity*
Con2. $(\Gamma \vdash A \text{ y } A \vdash B) \implies \Gamma \vdash B$ *Monotonicity*
Con3. $(\Delta \vdash A \text{ y } \Gamma, A \vdash B) \implies \Delta, \Gamma \vdash B$ *Transitivity*

So, a logic L is defined as a couple $\langle \text{For}, \vdash \rangle$ containing a consequence relation satisfying, on this paper, Con1, Con2 and Con3 and a set of formulas. We will say that Γ is a theory of L if $\Gamma \subseteq L$. We will also say that Γ is closed if it contains all of its consequences (the converse of Con1.)

For our purposes, For is a numerable set of symbols from the language that contains \neg as negation symbol, whether it is defined or native, besides some other logic symbols proper of each theory. Initially, as we have already said, we consider our logics with a consequence relation satisfying Con1, Con2 y Con3, but this does not mean that every consequence relation satisfies them. The following properties follow directly from Definition 1

Proposition 1. [2] *Let A, B be formulas and Δ, Γ be two theories of L. Then*

1. $(\Gamma, \Delta \not\vdash A) \implies \Gamma \not\vdash A$;
2. $(\Gamma \vdash A \text{ y } A \vdash B) \implies \Gamma \vdash B$;

$$3. (\Gamma \vdash A \text{ y } \Gamma, A \vdash B) \implies \Gamma \vdash B.$$

Definition 2. 1. We will say that a theory Γ is contradictory, with respect to \neg , if there exists a formula A such that $\Gamma \vdash A$ y $\Gamma \vdash \neg A$;

2. We say that a theory Γ is trivial if $\forall A : \Gamma \vdash A$;

3. We say that a theory is explosive if, when adding to it any couple of contradictory formulas, the theory becomes trivial;

4. We say that a logic L is contradictory (trivial, explosive) if all of its theories are contradictory (trivial, explosive).

The empty theory will be required due to the next property.

Proposition 2. [2] A logic L is contradictory (trivial, explosive) if and only if the empty theory is contradictory (trivial, explosive).

On Definition 2 they were established the concepts of a logic being contradictory, trivial or explosive; It is natural to ask whether all logics can be classified among these three types of logics. The following principles give an affirmative answer to this question.

Principle of non-contradiction (PNC)

$$L \text{ is not contradictory. } \exists \Gamma : \forall A : (\Gamma \not\vdash A \text{ or } \Gamma \not\vdash \neg A)$$

Principle of non-triviality (PNT)

$$L \text{ is not trivial. } \exists \Gamma : \exists B : (\Gamma \not\vdash B)$$

Principle of explosion or Principle of pseudo-scotus (PPS)

$$L \text{ is explosive. } \forall \Gamma : \forall A : \forall B : (\Gamma, A, \neg A \vdash B)$$

1.1 PARACONSISTENCY

Newton da Costa in [5] and S. Jaśkowski in [6], founders of the paraconsistent logic, propose to study logics in which one can introduce non-trivial contradictory theories. The result is contained in two definitions: The first one is based on the existence of non-trivial contradictory theories.

We say that the logic L is paraconsistent if

$$1.1 \quad \exists \Gamma : \exists A : \exists B : (\Gamma \vdash A, \Gamma \vdash \neg A \text{ y } \Gamma \not\vdash B)$$

The second definition is based on the existence of non-explosive theories.

We say that the logic L is paraconsistent if

$$1.2 \quad \exists \Gamma : \exists A : \exists B : (\Gamma, A, \neg A \not\vdash B)$$

From Definition 1.1, the concept of paraconsistency does not reject the Principle of Non-Contradiction, but, from Definition 1.2, will reject the Principle of Explosion.

Proposition 3. [2] *Definitions 1.1 and 1.2 are equivalent.*

Definition 3. [2]

1. We will say that the formulas A, B are equivalent if one implies the other and viceversa, that is: $A \vdash B$ and $B \vdash A$;

2. In a similar way, the theories Γ and Δ are equivalent if:

$$(\forall A \in \Gamma : \Delta \vdash A) \text{ y } (\forall B \in \Delta : \Gamma \vdash B).$$

From the previous definition, we obtain the following property.

Proposition 4. [2] *Let L be a paraconsistent logic. Then not all of its contradictions are equivalent.*

To assume the existence of paraconsistent logics implies assuming the existence of some non-explosive theories. In particular, we do not want the empty theory to be explosive, since it will be convenient to use it in some theoretical proofs.

On the other hand, the triviality is of importance for the study of paraconsistency, since it is of interest to know to what extent a logic can be turned trivial.

Proposition 5. [2] *If a logic is explosive, then it is finitely trivializable; this means, it has a finite trivial theory.*

Let us take Proposition 5 to the extreme of having a unique formula which trivializes any theory.

Definition 4. *We say that the logic L has a bottom particle if there exists a formula $C \in L$ such that it, by itself, makes L trivial. That is;*

$$\exists C : \forall \Gamma : \forall B : (\Gamma, C \vdash B)$$

In the case that such a formula exists, we denote it by \perp . Evidently, every logic having a bottom particle as one of its thesis, becomes trivial. We have a reformulation of The Principle of explosion .

Principle of ex falso

L has a bottom particle

Later we will see that the ex-contradictions are not necessarily the same as the ex falso.

Definition 5. We say that a logic has a top particle if there exists a formula C , in L , that is consequence of each of its theories, that is:

$$\exists C : \forall \Gamma : (\Gamma \vdash C)$$

In case it exists, we denote the top particle by \top . Any thesis of a logic is a top particle. It is easy to see that the addition of a top particle to a given theory is pretty innocuous.

Definition 6. We say that the logic L has strong negation (or supplementary negation) if there exists a scheme $\mu(A)$, that depends only upon A , such that:

a) $\exists A : \mu(A)$ is not a bottom particle

b) $\forall A : \forall \Gamma : \forall B : (\Gamma, A, \mu(A) \vdash B)$

We will denote the strong negation by $\mu(A) = \sim A$. The parallel definition of contradictory theory with respect to \sim becomes clear, furthermore we have the supplementary version of PPS.

Principle of ex contradiction (sPPS)

L has strong negation

Proposition 6. [2]

1. If a logic has a bottom particle or it has strong negation, then it is finitely trivializable;
2. If a non-trivial logic has the bottom particle, then it accepts strong negation;
3. If a logic is explosive and non-trivial, then it is supplementarily explosive (it does satisfies ex falso).

But, we would also like to recognize the form taken by \perp and \sim , as well as some reciprocal versions of the assertions in 6. In order to do that we need to recognize other connectives that will be very helpful.

Definition 7. [2] A logic L is said to be left-adjunctive if for any two formulas A and B there is a schema $\mu(A, B)$, depending only on A and B , with the following behavior:

a) $\exists A : \exists B : \mu(A, B)$ is not a bottom particle, and

b) $\forall A : \forall B : \forall \Gamma : \forall D : (\Gamma, A, B \vdash D \implies \Gamma, \mu(A, B) \vdash D)$.

Such a formula, in case it exists, will be denoted by $\mu(A, B) = A \wedge B$.

In order to characterize the left-adjunctive logics, we have the following proposition.

Proposition 7. [2] Let L be a logic. A conjunction in L is left-adjunctive if and only if it respects PC1:

a) $\exists A : \exists B : A \wedge B$ is not a bottom particle, and

b) $\forall \Gamma : \forall A : \forall B : (\Gamma, A \wedge B \vdash A \text{ and } \Gamma, A \wedge B \vdash B)$.

We have the following properties.

Proposition 8. [2] Let L be a left-adjunctive logic. Then

1. If L is finitely trivializable or if L has strong negation, then it has a bottom particle;
2. If L is finitely trivializable, then it inherits the explosive supplement.
3. If L respects ex contradiction, then it respects ex Falso.

2 ANALYSIS OF SOME LOGICS

In what follows we will give a diagnostic from paraconsistency in some logics. We start with an already known logic called Pac, which is treated in [2].

2.1 The Pac logic

Pac is determined by Table 1, the designated values are 1 y $\frac{1}{2}$. Observe that the connectives \vee and \wedge are determined by the functions *max* and *min*, respectively. The consequence relation is determined by:

$\Gamma \vdash B$ if and only if for any valuation ν such that $\nu(A)$ is a designated value, for every formula $A \in \Gamma$, $\nu(B)$ is also a designated value.

\wedge	0	$\frac{1}{2}$	1	\vee	0	$\frac{1}{2}$	1	\longrightarrow	0	$\frac{1}{2}$	1	\neg
0	0	0	0	0	0	$\frac{1}{2}$	1	0	1	1	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1	1	1	1	1	1	0	$\frac{1}{2}$	1	0

Table 1. Pac's semantics

- Taking ν, A, B such that $\nu(A) = \frac{1}{2}$, $\nu(B) = 0$. We have that $A, \neg A \not\vdash B$. Then, Pac is paraconsistent.
- By looking at the tables, we conclude the non-existence of a schema $\mu(A)$, that depends only upon A , such that $A, \mu(A) \vdash B$, for every B . Since taking $\nu(A) = \frac{1}{2}$ and $\nu(B) = 0$, then $\nu(\mu(A)) = \frac{1}{2}$, therefore Pac does not accept strong negation.
- By proposition 6, Pac does not admit a bottom particle
- It can be verified, by means of Proposition 7, that Pac is left-adjunctive.
- By proposition 8, Pac is not finitely trivializable.

2.2 The Lukasiewicz logic

The semantics of Lukasiewicz logic is determined by Table 2.

\longrightarrow	0	$\frac{1}{2}$	1	\neg
0	1	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1	1

Table 2. Lukasiewics semantics

The conjunction and disjunction connectives are determined by $\vee = \max$, and $\wedge = \min$. The designated value is 1, and the consequence relation is given by:

$$\Gamma \vdash_{Luk} A \implies \begin{cases} \nu(A) = 1 \\ \nu(B) = 0 & \text{for some } B \in \Gamma \\ \nu(B) = \frac{1}{2} = A & \text{for some } B \in \Gamma. \end{cases}$$

- Taking ν such that: $\nu(A) = \frac{1}{2}$ and $\nu(B) = 0$, we have $A, \neg A \not\vdash_{Luk} B$. Hence Luk is paraconsistent.
- Observe, from Table 2, that $\nu(\neg(A \rightarrow A)) = 0$, for all valuation ν . Then we define the bottom particle as $\perp := \neg(A \rightarrow A)$
- By proposition 6, Luk admits strong negation. And for it, we propose $\sim A := \neg((\neg A \vee A) \rightarrow A)$.
- Luk is left-adjunctive.
- By proposition 6, Luk es finitely trivializable.

2.3 The Logic RM_3

The semantics of RM_3 is determined by the functions min and max, and the next table.

\rightarrow	0	$\frac{1}{2}$	1	\neg
0	1	1	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	0	1	0

The designated values are 1 and $\frac{1}{2}$, and the consequence relation is given by:

$$\Gamma \vdash_{RM_3} B \iff \begin{cases} \nu(B) & \text{has the designated value} \\ \nu(A) = 0 & \text{for some } A \in \Gamma \\ \nu(A) = \frac{1}{2} = \nu(B) & \text{for some } A \in \Gamma. \end{cases}$$

- RM_3 is paraconsistent, taking $\nu(A) = \frac{1}{2}$ and $\nu(B) = 0$, we conclude that $A, \neg A \not\vdash B$.
- RM_3 is left-adjunctive.
- Observe that any scheme $\mu(A)$, that depends only on A , is such that if $\nu(A) = \frac{1}{2}$, then $\nu(\mu(A)) = \frac{1}{2}$. Then RM_3 does not admit strong negation.
- By 6, RM_3 does not admit a bottom particle.
- By 8 RM_3 is not finitely trivializable.

2.4 The \wp -Four logic

The semantics of \wp -Four is determined by the following tables:

A	$\Box A$	\longrightarrow	0	1	2	3	\neg_F	\wedge_F	0	1	2	3	\vee_F	0	1	2	3
0	0	0	3	3	3	3	3	0	0	0	0	0	0	0	1	2	3
1	0	1	2	3	2	3	3	1	0	1	0	1	1	1	1	3	3
2	0	2	1	1	3	3	3	2	0	0	2	2	2	2	3	2	3
3	3	3	0	1	2	3	0	3	0	1	2	3	3	3	3	3	3

Table 3. The \wp -Four semantics

The designated value is 3. In this case, the functions \min , \max do not coincide with the connectives \wedge_F and \vee_F ; We have, now, two new functions associated to \wedge_F y \vee_F , we will call them \min_F and \max_F , respectively. So, the consequence relation is given by:

$$\Gamma \vdash_F B \iff \begin{cases} \nu(B) = 3 \\ \min_F \{A \in \Gamma\} = 0 \\ \nu(B) = 2 \text{ y } \min_F \{A \in \Gamma\} = 2 \\ \nu(B) = 1 \text{ y } \min_F \{A \in \Gamma\} = 1 \end{cases}$$

- \wp -Four is paraconsistent; taking $\nu(A) = 1$ and $\nu(B) = 0$, we have that $A, \neg_F A \not\vdash B$.
- For every valuation ν , we have $\nu(\Box A \wedge_F \neg_F A) = 0$. Then \wp -Four accepts a bottom particle ($\perp := \Box A \wedge_F \neg_F A$).

- \wp -Four is left-adjunctive.
- Therefore \wp -Four accepts strong negation.
- \wp -Four is finitely trivializable.

2.5 The G'_3 logic

The G'_3 semantics is determined by Tables 4, and the functions $\wedge := \min$ y
 $\vee := \max$.

\longrightarrow	0	$\frac{1}{2}$	1	$\neg_{G'_3}$
0	1	1	1	1
$\frac{1}{2}$	0	1	1	1
1	0	$\frac{1}{2}$	1	0

Table 4. G'_3 semantics

The designated value is 1, and the consequence relation is determined by:

$$\Gamma \vdash B \iff \begin{cases} \nu(B) = 1 \\ \nu(A) = 0, \text{ for some } A \in \Gamma \\ \nu(A) = \frac{1}{2} = \nu(B), \text{ for some } A \in \Gamma \end{cases}$$

- Taking $\nu(A) = \frac{1}{2}$ y $\nu(B) = 0$, we have $A, \neg_{G'_3} A \not\vdash B$. then G'_3 is paraconsistent.
- G'_3 is left-adjunctive.
- G'_3 has the bottom particle; $\perp := \neg_{G'_3} A \wedge \neg_{G'_3} \neg_{G'_3} A$
- Then G'_3 has strong negation: ($\sim A := A \longrightarrow (\neg_{G'_3} A \wedge \neg_{G'_3} \neg_{G'_3} A)$)
- G'_3 es finitely trivializable.

An important property of G'_3 is that it has the same expressive power than Luk , this follows from the fact that negation and implication in Luk can be expressed in terms of negation and implication in G'_3 .

$$\neg_{Luk}A := [A \longrightarrow_{G'_3} (A \wedge \neg_{G'_3}A)] \wedge [A \vee (A \longrightarrow_{G'_3} \neg_{G'_3} \neg_{G'_3}A)]$$

$$A \longrightarrow_{Luk} B := (A \wedge \neg_{G'_3}A) \vee (A \longrightarrow_{G'_3} B)$$

An axiomatization of G'_3 is presented by Osorio et al in [4]

2.6 The C_ω logic

For this logic, there is no n -valuation that defines its semantics, hence we present the system C_ω by means of a Hilbert-style axiomatization. The axioms of C_ω are:

$$\text{Pos1. } A \longrightarrow (B \rightarrow A)$$

$$\text{Pos2. } (A \rightarrow (B \rightarrow C)) \longrightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

$$\text{Pos3. } (A \wedge B) \longrightarrow A$$

$$\text{Pos4. } (A \wedge B) \longrightarrow B$$

$$\text{Pos5. } A \longrightarrow (B \rightarrow A \wedge B)$$

$$\text{Pos6. } A \longrightarrow (A \vee B)$$

$$\text{Pos7. } B \longrightarrow (A \vee B)$$

$$\text{Pos8. } (A \rightarrow C) \longrightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$$

$$C_\omega 1. A \vee \neg A$$

$$C_\omega 2. \neg \neg A \longrightarrow A$$

The only inference rule is Modus Ponens (MP). The consequence relation is given in proof theory, that is to say:

$$\Gamma \vdash B \iff \left\{ \begin{array}{l} \text{there exist } B_1, \dots, B_n \text{ formulae such that} \\ 1. B_n = B \\ 2. \text{ For each } i = 1, \dots, n \text{ we have that} \\ \quad 2.1 } B_i \in \Gamma \text{ or} \\ \quad 2.2 } B_i \text{ is an atom or} \\ \quad 2.3 } B_i \text{ is direct consequence of precedents } B_j, B_k \text{ under MP.} \end{array} \right.$$

– C_ω is left-adjunctive, due to Pos3 y Pos4.

In [2], we can see, in particular, the following results:

- Pierce rule is not valid in C_ω , in other words: $\not\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$.
 - The Deduction Theorem is valid in C_ω .
 - C_ω is not finitely trivializable.
- C_ω does not have a bottom particle (Proposition 6)
 - C_ω does not have strong negation (Proposition 8)

3 CONCLUSION

The present work is just an initial analysis in which, first, we detect the paraconsistent logics, then we recognize their properties like being left-adjunctive, having the bottom particle and strong negation, and we determined whether they are finitely trivializable. All logics in this survey were paraconsistent. For future work, it remains to analyze the concept of a theory being (finitely) *gently explosive*, *partially explosive* and *controllably explosive*.

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