Handling Imprecise Knowledge with Fuzzy Description Logic

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Abstract

Fuzzy Description Logics have been proposed in the literature as a way to represent and reason with vague and imprecise knowledge. Their decidability, the empirically tractable and efficient reasoning algorithms, that carry over to fuzzy Description Logics, have attracted the attention of many research communities and domains that deal with a wealth of imprecise knowledge and information. In the current paper we present the syntax and semantics of fuzzy $\mathcal{SHOIQ}$, investigating several properties of the semantics of transitivity, qualified cardinality restrictions and reasoning capabilities.

1 Introduction

Although Description Logics (DLs) provide considerable expressive power, they feature expressive limitations regarding their ability to represent vague and imprecise knowledge. Consider for example an image processing application. Such applications can be assisted by the aid of a knowledge base that contains definitions of the objects that can be found within an image. For example there could be a definition of the form,

$$\mathsf{Body} \sqcap \exists \mathsf{hasPart}.\mathsf{Tail} \sqsubseteq \mathsf{Animal}.$$

saying that if an object has a body and a part that is a tail then this object is an animal. Now suppose that we run an image analysis algorithm. Such algorithms usually segment and label objects that they identify in images. Since the algorithms cannot be certain about the membership or non-membership of an object to a concept it usually assigns degrees of truth to these labellings. For example, we could have that the object o_1 hasPart o_2 to a degree of 0.8, that o_1 is a Body to a degree of 0.6 and that o_2 is a Tail to a degree of 0.7. From this knowledge we can deduce that o_1 is an Animal to a degree, at-least equal to 0.6. For that purpose *Fuzzy Description Logics* (f-DLs), have been proposed in the literature as a way to represent and reason with vague and imprecise knowledge. In the current paper we present the f-DL, f-$\mathcal{SHOIQ}$. f-$\mathcal{SHOIQ}$ extends f-$\mathcal{SHOIN}$ [10] with qualified cardinality restrictions (QCRs). Furthermore, we investigate the semantics of f-$\mathcal{SHOIQ}$, showing that it is a sound extension of $\mathcal{SHOIQ}$, we investigate properties of fuzzy QCRs and transitivity, we provide the inference problems and investigate reasoning capabilities in f-$\mathcal{SHOIQ}$.

2 Fuzzy Set Theory

Fuzzy set theory and fuzzy logic are widely used for capturing imprecise knowledge [3]. While in classical set theory an element either belongs to a set or not, in fuzzy set theory elements belong only to a certain degree. More formally, let X be a set of elements. A fuzzy subset A of X, is defined by a *membership function* $\mu_A(x)$, or simply $A(x)$ [3]. This function assigns any $x \in X$ to a value between 0 and 1 that represents the degree in which this element belongs to X. In this new framework the classical set theoretic and logical operations are performed by special mathematical functions. More precisely *fuzzy complement* is a unary operation of the form $c : [0,1] \rightarrow [0,1]$, *fuzzy intersection* and *union* are performed by two binary functions of the form $t : [0,1] \times [0,1] \rightarrow [0,1]$ and $u : [0,1] \times [0,1] \rightarrow [0,1]$, called *t-norm* and *t-conorm* operations [3], respectively, and *fuzzy implication* also by a binary function, $\mathcal{J} : [0,1] \times [0,1] \rightarrow [0,1]$. In order to produce meaningfull fuzzy complements, conjunctions, disjunctions and implications, these functions must satisfy certain mathematical properties. For example the operators must satisfy the following boundary properties, $c(0) = 1$, $c(1) = 0$, $t(1, a) = a$ and $u(0, a) = a$. Due to space limitations we cannot present all the properties that these functions should satisfy. The reader is referred to [3] for a comprehensive introduction. Nevertheless, it worths noting here that there exist two distinct classes of fuzzy implications, those of S-implications, given by the equation $\mathcal{J}(a, b) = u(c(a), b)$, and those of R-implications, given by $\mathcal{J}(a, b) = \sup\{x \in [0,1] \mid t(a, x) \leq b\}$. Examples of fuzzy operators are the Lukasiewicz negation, $c_L(a) = 1 - a$, t-norm, $t_L(a, b) = \max(0, a + b - 1)$, t-conorm $u_L(a, b) = \min(1, a + b)$, and implication, $\mathcal{J}_L(a, b) = \min(1, 1 - a + b)$, the Gödel norms $t_G(a, b) = \min(a, b)$, $u_G(a, b) = \max(a, b)$, and implication $\mathcal{J}_G(a, b) = b$ if $a > b$, 1 otherwise, and the Kleene-Dienes implication (KD-implication), $\mathcal{J}_{KD}(a, b) = \max(1 - a, b)$.

3 Fuzzy Description Logics

3.1 Syntax and Semantics

In this section we introduce the DL f-$\mathcal{SHOIQ}$. As usual we have an alphabet of distinct concept names ($\mathbf{C}$), role names ($\mathbf{R}$) and individual names ($\mathbf{I}$). f-$\mathcal{SHOIQ}$-roles and f-$\mathcal{SHOIQ}$-concepts are defined as follows:

Definition 3.1 *Let $RN \in \mathbf{R}$ be a role name and R an f-$\mathcal{SHOIQ}$-role. f-$\mathcal{SHOIQ}$-roles are defined by the abstract syntax: $R ::= RN \mid R^-$. The inverse relation of roles is symmetric, and to avoid considering roles such as R^{--}, we define a function* Inv*, which returns the inverse of a role, more precisely $\mathsf{Inv}(R) := R^-$ and $\mathsf{Inv}(R^-) := R$. Let $A \in \mathbf{C}$ be a concept name, C and D f-$\mathcal{SHOIQ}$-concepts, $p \in \mathbb{N}$, S a* simple[1] *f-$\mathcal{SHOIQ}$-role, $o \in \mathbf{I}$ and R an f-$\mathcal{SHOIQ}$-role. f-$\mathcal{SHOIQ}$ concepts are defined by the following abstract syntax:*

$$C, D \longrightarrow \bot \mid \top \mid A \mid \neg C \mid C \sqcup D \mid C \sqcap D \mid \exists R.C \mid \forall R.C \mid \leq pS.C \mid \geq pS.C \mid \{o\}$$

The semantics of f-DLs are provided by a *fuzzy interpretation* [9]. A fuzzy interpretation is a pair $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ where the domain $\Delta^{\mathcal{I}}$ is a non-empty set of objects, called the *domain of interpretation*, and $\cdot^{\mathcal{I}}$ is a *fuzzy interpretation function* which maps an individual $\mathsf{a} \in \mathbf{I}$ to an element $\mathsf{a}^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, a concept name $\mathsf{A} \in \mathbf{C}$ to a membership function $\mathsf{A}^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$ and a role name $R \in \mathbf{R}$ to a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$. The semantics of f-$\mathcal{SHOIQ}$-concepts and roles are depicted in Table 1. Most of these semantics have been presented elsewhere [9, 10, 1, 4, 7]. Nevertheless, the semantics of fuzzy QCRs presented here are a revision of the definition provided in [10]. First, we extend the semantics of number restriction to qualified number restrictions. Second, we use arbitrary fuzzy implications to give semantics to at-most QCRs, while in [10] only S-implications were considered. Extending the definition has the effect that if $\forall R.C$ is interpreted by means of an R-implication and the fuzzy complement is the *involutive* $(c(c(a)) = a)$ *precomplement* of the R-implication, then the equivalence $\forall R.C \equiv \leq 0R.\neg C$, holds. Since we are using arbitrary fuzzy implications we have to extend the definition in [10] to consider the equalities $(=)$ and inequalities $(\neq)$ of objects. Please note that this equality and inequality is usually considered crisp, i.e. either 0 or 1, in the f-DL literature [5]. Finally, as it is argued in [7] we choose not to fuzzify nominal concepts. The reason for this choice is that a concept of the form $\{\mathrm{o}\}$ intends to refer to a specific object of $\Delta^{\mathcal{I}}$, i.e. $o^{\mathcal{I}}$ and not some real life concept with an arbitrary number of members.

An f-$\mathcal{SHOIQ}$ knowledge base Σ consists of a TBox, an RBox and an ABox. Let C and D be f-$\mathcal{SHOIQ}$ concepts. As with the classical case, an f-$\mathcal{SHOIQ}$

[1] A role is called *simple* if it is neither transitive nor has any transitive sub-roles.

Table 1: Semantics of f-$\mathcal{SHOIQ}$-concepts and roles

Constructor	Syntax	Semantics
top	$\top$	$\top^{\mathcal{I}}(a) = 1$
bottom	$\bot$	$\bot^{\mathcal{I}}(a) = 0$
general negation	$\neg C$	$(\neg C)^{\mathcal{I}}(a) = c(C^{\mathcal{I}}(a))$
conjunction	$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}}(a) = t(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
disjunction	$C \sqcup D$	$(C \sqcup D)^{\mathcal{I}}(a) = u(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
exists restriction	$\exists R.C$	$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} \{t(R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))\}$
value restriction	$\forall R.C$	$(\forall R.C)^{\mathcal{I}}(a) = \inf_{b \in \Delta^{\mathcal{I}}} \{\mathcal{J}(R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))\}$
nominal	$\{\mathsf{o}\}$	$\{\mathsf{o}\}^{\mathcal{I}}(a) = 1$ if $a \in \{\mathsf{o}^{\mathcal{I}}\}$, otherwise $\{\mathsf{o}\}^{\mathcal{I}}(a) = 0$
at-most QCR	$\leq pR.C$	$\inf_{b_1,\ldots,b_{p+1} \in \Delta^{\mathcal{I}}} \mathcal{J}(t_{i=1}^{p+1}\{t(R^{\mathcal{I}}(a,b_i), C^{\mathcal{I}}(b_i))\}, u_{i<j}\{b_i = b_j\})$
at-least QCR	$\geq pR.C$	$\sup_{b_1,\ldots,b_p \in \Delta^{\mathcal{I}}} t(t_{i=1}^{p}\{t(R^{\mathcal{I}}(a,b_i), C^{\mathcal{I}}(b_i))\}, t_{i<j}\{b_i \neq b_j\})$
inverse roles	R^-	$(R^-)^{\mathcal{I}}(b,a) = R^{\mathcal{I}}(a,b)$

TBox is a finite set of axioms of the form $C \sqsubseteq D$, which are called, *fuzzy inclusion axioms*. A fuzzy interpretation $\mathcal{I}$ satisfies a fuzzy TBox $\mathcal{T}$ if $\forall o \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(o) \leq D^{\mathcal{I}}(o)$, for each $C \sqsubseteq D \in \mathcal{T}$; in this case, we say that $\mathcal{I}$ is a *model* of $\mathcal{T}$. Similarly, an f-$\mathcal{SHOIQ}$ RBox is a finite set of *fuzzy transitive role* axioms, $\mathsf{Trans}(R)$, and *fuzzy role inclusion* axioms, $R \sqsubseteq S$. $\mathcal{I}$ satisfies a fuzzy RBox $\mathcal{R}$ if $\forall a, c \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(a,c) \geq \sup_{b \in \Delta^{\mathcal{I}}}\{t(R^{\mathcal{I}}(a,b), R^{\mathcal{I}}(b,c))\}$ for each $\mathsf{Trans}(R) \in \mathcal{R}$ and $\forall a, b \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,b) \leq S^{\mathcal{I}}(a,b)$, for each $R \sqsubseteq S \in \mathcal{T}$; in this case, we say that $\mathcal{I}$ is a model of $\mathcal{R}$. An f-$\mathcal{SHOIQ}$ ABox is a finite set of fuzzy assertions [9] of the form $(\mathsf{a} : C) \bowtie n$ or $(\langle \mathsf{a}, \mathsf{b} \rangle : R) \bowtie n$, where $\bowtie$ stands for $\geq, >, \leq$ and $<$. Formally, $\mathcal{I}$ satisfies a fuzzy ABox $\mathcal{A}$, if $C^{\mathcal{I}}(\mathsf{a}^{\mathcal{I}}) \geq n$ ($R^{\mathcal{I}}(\mathsf{a}^{\mathcal{I}}, \mathsf{b}^{\mathcal{I}}) \geq n$) for each $(\mathsf{a} : C) \geq n$ ($(\langle \mathsf{a}, \mathsf{b} \rangle : R) \geq n$) in $\mathcal{A}$; in this case, we say that $\mathcal{I}$ is a model of $\mathcal{A}$. The satisfiability of fuzzy assertions with $\leq, >, <$ is defined analogously. Observe that we can also simulate assertions of the form $(\mathsf{a} : C) = n$ by considering the assertions $(\mathsf{a} : C) \geq n$ and $(\mathsf{a} : C) \leq n$.

As it has been argued in the literature, fuzzy set theory is an extension of classical set theory. Hence, the following lemma,

Lemma 3.2 *Fuzzy interpretations coincide with crisp interpretations if we restrict to the membership degrees of 0 and 1.*

In other words in the extreme limits of 0 and 1 the fuzzy operations provide the results of Boolean algebra. We call such an extention a *sound* extension.

Since we have defined fuzzy QCRs it is possible that $(a : (\geq p_1 R.C)) \geq n_1$ and $(a : (\leq p_2 R.C)) \geq n_2$, with $p_1 > p_2$ simultaneously hold, without forming a contradiction. More precisely if t is the Gödel t-norm and $\mathcal{J}$ the KD-implication we have,

Lemma 3.3 *Let* $\mathcal{A} = \{(a : (\geq p_1 R.C)) \geq n_1, (a : (\leq p_2 R.C)) \geq n_2\}$ *be a fuzzy ABox, with* $n_1, n_2 \in [0,1]$, $p_1, p_2 \in \mathbb{N}$, *and* $p_2 < p_1$. *Then* $\mathcal{A}$ *is satisfiable iff* $n_1 + n_2 \leq 1$.

In classical DLs, since $n_1, n_2 \in \{0, 1\}$, the inequality $n_1 + n_2 \leq 1$ is satisfied if and only if either $n_1 = 0$ or $n_2 = 0$. Indeed in crisp DLs an individual cannot simultaneously belong to both such concepts. Please note that investigating this property when other norm operations are used is an open research issue.

3.2 Logical Properties of Fuzzy DLs

As it is obvious different fuzzy operators specify different f-DLs. For example the f_{KD}-$\mathcal{SHOIQ}$ is obtained from f-$\mathcal{SHOIQ}$ when the Lukasiewicz negation, the Gödel t-norm and t-conorm and the Kleene-Dienes fuzzy implication are used, while f_L-$\mathcal{SHOIQ}$ is obtained if we use the Lukasiewicz negation, t-norm, t-conorm and fuzzy implication. The choice of the operations has an immediate impact on the logical properties of the resulting f-DL.

For any triple $\langle c, t, u \rangle$, due to the standard properties of the fuzzy operators [3], the following concept equivalences hold: $\neg\top \equiv \bot$, $\neg\bot \equiv \top$, $C \sqcap \top \equiv C$, $C \sqcup \bot \equiv C$, $C \sqcup \top \equiv \top$ and $C \sqcap \bot \equiv \bot$. If the complement is involutive it also holds that $\neg\neg C \equiv C$. Now if the fuzzy triple satisfies the De Morgan laws (called *dual* triple), we additionally have, $\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$ and $\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$. For example the fuzzy triples, $\langle c_L, t_L, u_L \rangle$ and $\langle c_L, t_G, u_G \rangle$, are dual triples. Moreover, for any dual triple $\langle c, t, u \rangle$ and S-implication $\mathcal{J}_S$ it holds that, $\forall R.C = \neg(\exists R.\neg C)$. For example the quadruple $\langle c_L, t_G, u_G, \mathcal{J}_{KD} \rangle$, satisfies this equivalence. Furthermore, if the fuzzy triple satisfies the laws of contradiction and excluded middle, then the following properties of boolean logic hold: $(C \sqcap \neg C \equiv \bot)$ and $(C \sqcup \neg C \equiv \top)$. For example, the triple $\langle c_L, t_L, u_L \rangle$, satisfies these laws.

It is important to notice that the classical properties of Boolean algebra, like the De Morgan the excluded middle and the contradiction laws, do not always hold in fuzzy set theory and logic. For example for the triple $\langle c_L, t_G, u_G \rangle$ we have $\max(0.6, 1 - 0.6) = 0.6 \neq 1$. The consequences of this property is that many well-known techniques in DLs, like *internalization* do not carry over to f-DLs. Fortunately, the laws of excluded middle and contradiction can be simulated.

Lemma 3.4 *For all $a \in \Delta^{\mathcal{I}}$, $n \in [0, 1]$, and interpretations $\mathcal{I}$*

1. *either $C^{\mathcal{I}}(a) < n$, or $C^{\mathcal{I}}(a) \geq n$ [8] [2], and*
2. *if $C^{\mathcal{I}}(a) \geq n$ and $(\neg C)^{\mathcal{I}}(a) \geq n$, then $\bot^{\mathcal{I}}(a) \geq \max(0, n - \epsilon)$, where ϵ is the* equilibrium *[3] of a fuzzy complement.*

Point 1 simulates the DL axiom $\top \sqsubseteq C \sqcup \neg C$, while point 2 the axiom $C \sqcap \neg C \sqsubseteq \bot$. For example for $\langle c_L, t_G, u_G \rangle$, where $\epsilon = 0.5$ if $n = 0.7$, then $0 = \bot^{\mathcal{I}}(a) \geq 0.2$, which is a contradiction, while for $n = 0.3$, $\bot^{\mathcal{I}}(a) \geq 0$, which is valid. Indeed it is possible that $C^{\mathcal{I}}(a) \geq 0.3$ and $(\neg C)^{\mathcal{I}}(a) \geq 0.3 = C^{\mathcal{I}}(a) \leq 0.7$.

[2] Similarly either $C^{\mathcal{I}}(a) \leq n$ or $C^{\mathcal{I}}(a) > n$ for all $n \in [0, 1]$

3.3 Inference Services

In the current section we will present the inference problems of f-DLs. An f-$\mathcal{SHOIQ}$ knowledge base Σ is *satisfiable* (*unsatisfiable*) iff there exists (does not exist) a fuzzy interpretation $\mathcal{I}$ which satisfies all axioms in Σ. An f-$\mathcal{SHOIQ}$-concept C is *n-satisfiable* w.r.t. Σ iff there exists a model $\mathcal{I}$ of Σ for which there is some $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = n$, and $n \in (0, 1]$; C subsumes D w.r.t. Σ iff for every model $\mathcal{I}$ of Σ we have $\forall d \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$; a fuzzy ABox $\mathcal{A}$ is *consistent* (*inconsistent*) w.r.t. a fuzzy TBox $\mathcal{T}$ and RBox $\mathcal{R}$ if there exists (does not exist) a model $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{R}$ that satisfies each assertion in $\mathcal{A}$. Given a fuzzy concept axiom, a fuzzy role axiom or a fuzzy assertion ϕ, Σ *entails* ϕ, written $\Sigma \models \phi$, iff for all models $\mathcal{I}$ of Σ, $\mathcal{I}$ satisfies ϕ.

Let $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$, be a fuzzy knowledge base. It has been proved that all inference problems of f-DLs can be reduced to ABox consistency w.r.t. $\mathcal{T}$ and $\mathcal{R}$. More precisely, C is n-satisfiable w.r.t. Σ iff $\langle \mathcal{T}, \mathcal{R}, \{(\mathtt{a} : C) \geq n\} \rangle$ is satisfiable, $\Sigma \models \phi \bowtie n$ iff $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{\phi \neg \bowtie n\} \rangle$ is unsatisfiable (where $\neg \bowtie$ represents the *negation* of inequalities, e.g. $\neg \geq = <$), and $\Sigma \models C \sqsubseteq D$ iff $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(\mathtt{a} : C) \geq n, (\mathtt{a} : D) < n\} \rangle$, for both $n \in \{n_1, n_2\}$, $n_1 \in (0, 0.5]$ and $n_2 \in (0.5, 1]$, is unsatisfiable [9]. In the past, the consistency problem in f-DLs has been considered w.r.t. to a simple and acyclic TBox. Only recently a procedure for deciding fuzzy ABox consistency w.r.t. general and/or cyclic TBoxes has been developed [8]. In classical DLs general and cyclic TBoxes were handled by a process called *internalization* [2]. As we mentioned previously, internalization is based on the law of excluded middle, which is not always satisfied in f-DLs. In [8] the authors use the case analysis of lemma 3.4, providing the following result,

Lemma 3.5 *[8] A fuzzy interpretation $\mathcal{I}$ satisfies $C \sqsubseteq D$ iff for all $n \in [0, 1]$ and $a \in \Delta^{\mathcal{I}}$, either $C^{\mathcal{I}}(a) < n$ or $D^{\mathcal{I}}(a) \geq n$.*

Hence, the semantic restrictions of a TBox $\mathcal{T}$ can be encoded in mutually exclusive fuzzy assertions.

4 Reasoning in Fuzzy Description Logics

As we have seen f-DLs constitute a sound extension of classical DLs. Hence, the techniques used to perform reasoning in classical DLs could be extended to provide reasoning support for f-DLs. Since all inference problems can be reduced to the problem of ABox consistency w.r.t. an RBox $\mathcal{R}$, a procedure that decides this problem should be constructed. In classical DLs this is done with the aid of tableaux algorithms that given an ABox $\mathcal{A}$ they try to construct a *tableau* for $\mathcal{A}$ [2], i.e., an abstraction of a model of $\mathcal{A}$ which has a tree or forest-like shape. In such trees, nodes correspond to objects in the model, and edges to certain relations that connect two nodes. Each node x is labelled with the set of concepts that it belongs to ($\mathcal{L}(x)$), and each edge $\langle x, y\rangle$ with a set of roles that connect two nodes x, y ($\mathcal{E}(\langle x, y\rangle)$). In the fuzzy case, since now we have fuzzy assertions, we extend these mappings to also include the membership degree that a node belongs to a concept. More formally we have the following definition.

Definition 4.1 *If $\mathcal{A}$ is a fuzzy ABox, $\mathcal{R}$ a fuzzy RBox, $\boldsymbol{R}_{\mathcal{A}}$ is the set of roles occurring in $\mathcal{A}$ and $\mathcal{R}$ together with their inverses, $sub(\mathcal{A})$ is the set of sub-concepts that exist in $\mathcal{A}$ and $\mathbf{I}_{\mathcal{A}}$ is the set of individuals in $\mathcal{A}$, a fuzzy tableau T for $\mathcal{A}$ w.r.t. $\mathcal{R}$, is defined to be a quadruple ($\boldsymbol{S}$, $\mathcal{L}$, $\mathcal{E}$, $\mathcal{V}$) such that: $\boldsymbol{S}$ is a set of elements, $\mathcal{L} : \boldsymbol{S} \times sub(\mathcal{A}) \rightarrow [0, 1]$ maps each element and concept to the membership degree of that element to the concept, $\mathcal{E} : \boldsymbol{R}_{\mathcal{A}} \times \boldsymbol{S} \times \boldsymbol{S} \rightarrow [0, 1]$ maps each role and pair of elements to the membership degree of the pair to the role, and $\mathcal{V} : \mathbf{I}_{\mathcal{A}} \rightarrow \boldsymbol{S}$ maps individuals occurring in $\mathcal{A}$ to elements in $\boldsymbol{S}$.*

Additionally, a fuzzy tableau should satisfy certain properties of the semantics of the f-DL language [6, 5]. For example, if concept conjunction is performed by the Gödel t-norm, then for $a \in \mathbf{S}$, $C, D \in sub(\mathcal{A})$, $n \in [0, 1]$, and $\mathcal{L}(s, C \sqcap D) \geq n$, it follows that both $\mathcal{L}(s, C) \geq n$ and $\mathcal{L}(s, D) \geq n$ must hold. Currently, we have tableau definitions for the languages f_{KD}-$\mathcal{SI}$ and f_{KD}-$\mathcal{SHIN}$ [6, 5], while it is an open research issue to define a tableau structure for f_{KD}-$\mathcal{SHOIQ}$ as well as for f-DLs with other norm operations. One difficult point in these definitions is to handle transitivity in the new framework. In classical DLs the tree-like structure is preserved by pushing concepts of the form $\forall R.C$ from a node s to a node t if $R(s, t)$ exists. This is based on the observation that if $\mathsf{Trans}(R)$ then the axiom $\forall R.C \sqsubseteq \forall R.(\forall R.C)$, holds [2]. The following lemma characterizes transitivity in f_{KD}-DLs.

Lemma 4.2 *If $(\forall R.C)^{\mathcal{I}}(a) \geq n$, $R^{\mathcal{I}}(a, b) \geq r_1$ and $\mathsf{Trans}(R)$ then, in an f_{KD}-DL, $(\forall R.(\forall R.C))^{\mathcal{I}}(a) \geq n$ holds.*

Thus, as a tableau property we have that, if $\mathcal{L}(s, \forall R.C) \geq n$ and $\mathsf{Trans}(R)$, then either $c(\mathcal{E}(R, \langle s, t\rangle)) \geq n$ or $\mathcal{L}(t, \forall R.C) \geq n$. Another major problem towards our goal in constructing a tableaux reasoning algorithm is to determine if the appropriate blocking techniques can be applied. While this is quite straightforward in f_{KD}-DLs [6, 5, 8], this is very hard when other norms are used.

5 Conclusions

Fuzzy Description Logics are applicable in a number of research and industrial applications that face a vast amount of imprecise and vague information. The last couple of years the work on fuzzy DLs has provided with impressive results such as the extension of very expressive DL languages, like $\mathcal{SHOIN}$ [10] and $\mathcal{SHOIQ}$, the development of reasoning procedures for fuzzy DLs like f_{KD}-$\mathcal{SI}$ [6] and f_{KD}-$\mathcal{SHIN}$ [5] and reasoning w.r.t. general inclusion axioms [8], which was considered an open problem for fuzzy DLs for many years. Currently the work on fuzzy DLs is focused on the reasoning problem for f_{KD}-$\mathcal{SHOIQ}$, on reasoning with other norms than the ones used in f_{KD}-$\mathcal{SI}$ and f_{KD}-$\mathcal{SHIN}$, on data-type support and on implementations.

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