Analytic Study of Opinion Dynamics in Multi-Agent Systems with Two Classes of Agents

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***Abstract*—This paper describes a model for opinion dynamics in multi-agent systems composed of two classes of agents. Each class is characterized by distinctive values of the parameters that govern opinion dynamics. The proposed model is inspired by kinetic theory of gases, according to which macroscopic properties of gases are described starting from microscopic interactions among molecules. By interpreting agents as molecules of gases, and their interactions as collisions among molecules, the equations that govern kinetic theory can be reinterpreted to model opinion dynamics in multi-agent systems. A key feature of the adopted kinetic-based approach is that it allows macroscopic properties of the system to be derived analytically. In order to take into account that the considered multi-agent system is composed of two classes of agents, kinetic theory of gas mixtures, which deals with gases composed of different kinds of molecules, is adopted. Presented results show that consensus is reached after a sufficiently large number of interactions, which depends on the parameters associated with the two classes of agents.**

I. Introduction

Opinion dynamics and consensus formation are well-known problems that deal with the identification of interaction rules which lead to proper distribution of opinion in multi-agent systems [1]. Such problems are important topics of the research on multi-agent systems and distributed computing and they have applications in many areas, such as control theory, physics, biology, and sociology (e.g., [2]). Various approaches have been proposed in the literature to describe opinion dynamics and consensus formation, among which we can recall those based on thermodynamics (e.g., [3]), on Bayesian networks (e.g., [4]), and on gossip-based algorithms (e.g., [4]). The use of cellular automata to model consensus formation has also been investigated; in this case, opinion is modeled as a discrete variable and consensus is reached through proper transition rules (e.g., [5]). Another important framework which is useful to study opinion dynamics is related to graph theory. (e.g., [6]).

In this paper, we consider a model for opinion dynamics which is inspired by *sociophysics*, a discipline according to which social interactions and opinion dynamics in multi-agent systems can be described using the formalism of the kinetic theory of gases [7]. Kinetic theory of gases aims at analyzing the effects of microscopic collisions among the molecules from a probabilistic point of view in order to derive macroscopic properties of gases by means of a proper balance equation, namely, the Boltzmann equation [8]. According to sociophysics, a parallelism can be done between the molecules of gases and agents in multi-agent systems: collisions among the molecules are reinterpreted as interactions among agents. A major advantage of the use of kinetic-based approaches to model opinion dynamics and consensus is that analytic results can be derived, while, at the opposite, opinion dynamics in multi-agent systems is typically investigated through simulations [9]. It is worth noting that common approaches to the analysis of interactions in multi-agent systems (e.g., [10]) are normally more interested in formalizing complex microscopic interactions rather than in studying the overall emergent behavior of the system.

Standard kinetic theory typically assumes that all the molecules are equal. However, gases are typically composed of molecules of different types and, therefore, a more accurate description of gases can be achieved using kinetic theory of *gas mixtures*, which takes into account that different species of molecules coexist in the same gas. Using the framework of kinetic theory of gas mixtures, it is then possible to describe multi-agent systems composed of different classes of agents, each of which is associated with different values of the parameters used to model opinion dynamics. The most important features that can be introduced to distinguish a specific type of agents are the propensity to change opinion when interacting with other agents, and the ability to change the opinions of interacting agents. In addition, different classes of agents can have different cardinalities and different initial distribution of opinions. According to this approach, phenomena such as *extremism* or *skepticism* can be studied [6]. In this paper, we consider multi-agent systems composed of two classes of agents; however, the proposed approach is general and, potentially, the number of classes of agents can be set equal to the number of agents, thus having one agent for each class.

This paper is organized as follows. In Section II the opinion dynamics problem is formulated using the kinetic framework. In Section III macroscopic properties of the considered multi-agent system are derived. Section IV shows some illustrative results concerning the average opinions of specific multi-agent systems. Section V concludes the paper.

II. Kinetic Formulation of Opinion Formation

The identification of agents with the molecules of a gas allows applying the framework of kinetic theory to different fields and, in particular, to distributed artificial intelligence and opinion dynamics in multi-agent systems. As in kinetic theory, we assume that each agent can interact with any other agent in the system and that each interaction involves two agents [11]. For this reason, we denote interactions as *binary*. While the molecules of gases are typically related to their velocities, in the context of opinion dynamics we assume that each agent is associated with a scalar attribute v that denotes its opinion. The opinion of each agent is updated at each interaction, according to proper rules. Various kinds of rules to update the opinion of agents after interactions have been studied to model different characteristics of agents [12], [13].

Let us denote as $n(t)$ the total number of agents at time t and as $n_1(t)$ and $n_2(t)$ the number of agents of the type 1 and 2, respectively, so that $n(t) = n_1(t) + n_2(t)$. With no loss of generality, we assume in the rest of this paper that the opinion of each agent is defined in the interval $I = [-1, 1]$, where -1 and 1 represent extremal opinions. The considered model is aimed at describing the temporal evolution of the opinion by studying the effects of pairwise interactions.

A. Interaction Rules

In order to describe the microscopic effects of pairwise interactions, let us define the interaction rules. Assume that an agent of type s with opinion v interacts with another agent of type r with opinion w. The post-interaction opinions of the two interacting agents depend on their pre-interaction opinions according to the following rules

$$\begin{cases} v^* = v - \gamma_{sr}(v - w) \\ w^* = w - \gamma_{rs}(w - v) \end{cases} \tag{1}$$

where v^* and w^* are the opinions of the two agents after the interaction. Observe that the considered model involves 4 coefficients $\{\gamma_{sr}\}_{s,r=1}^2$, where γ_{sr} measures the propensity of an agent of type s to change its opinion in favor of that of an agent of type r. As a matter of fact, considering, for instance, the first equation of (1) it is clear that an increment of γ_{sr} increases the propensity of agents of type s to change their opinions when interacting with agents of type r. In the following, we assume that the coefficients $\{\gamma_{sr}\}_{s,r=1}^2$ satisfy

$$0 < \gamma_{sr} < \frac{1}{2} \qquad \forall\, s, r \in \{1, 2\}. \tag{2}$$

In agreement with the intended meaning of γ_{sr} explained above, according to (1), if γ_{sr} is nearly 0, the individuals of type s are not inclined to change their opinion towards that of agents of type r. For this reason, values of γ_{sr} close to 0 characterize skeptical agents. At the opposite, if in the first equation of (1) we set $\gamma_{sr} \simeq 1/2$, then $v^* \simeq 1/2(v + w)$, so that the first agent looses half of its opinion in favour of that of the second, which characterize easily influenced agents.

The sum of the opinions of two interacting agents after the interaction can be derived from (1) and it is given by

$$v^* + w^* = v + w + (\gamma_{rs} - \gamma_{sr})(v - w). \tag{3}$$

From (3), the opinion is not conserved and that it can change depending on the sign of $(\gamma_{rs} - \gamma_{sr})(v - w)$, namely on the values of the coefficients γ_{rs} and γ_{sr} and on the values of the pre-interaction opinions v and w. From (1) it can also be derived that the difference of the opinions of two interacting agents after the interaction is

$$v^* - w^* = \varepsilon_{rs}(v - w). \tag{4}$$

where $\varepsilon_{rs} = 1 - (\gamma_{rs} + \gamma_{sr})$. Since, from (2), $\gamma_{rs} + \gamma_{sr} \in (0, 1)$, it is easy to conclude that $\varepsilon_{rs} \in (0, 1)$. Therefore, from (4) we can conclude that the difference between the post-interaction opinions is smaller than the difference between the pre-interaction opinions of the two agents. Hence, it is reasonable to expect that, after a sufficiently large number of interactions, all agents end up with the same opinion, regardless of their class. Concerning differences of opinions, it can also be concluded that the post-interaction opinion of an agent is closer to its pre-interaction opinion than to the pre-interaction opinion of the agent it interacts with. As a matter of fact, from (2) and (1), one can derive that

$$\begin{aligned} |v^* - v| &= \gamma_{sr}|v - w| < (1 - \gamma_{sr})|v - w| = |v^* - w| \\ |w^* - w| &= \gamma_{rs}|w - v| < (1 - \gamma_{rs})|w - v| = |w^* - v|. \end{aligned} \tag{5}$$

We remark that, according to the model in (1), the post-interaction opinions v^* and w^* still belong to the interval I where the opinions are defined.

B. The Boltzmann Equation

Starting from the interaction rules in (1), it is possible to study opinion dynamics of multi-agent systems using simulations. Instead, we now show how to obtain analytical results by applying the framework of kinetic theory of gas mixtures to the considered opinion dynamics scenario. For this purpose, we introduce the Boltzmann equation, namely an integro-differential equation that allows deriving macroscopic properties of gases. In the considered scenario, which includes only two classes of agents, two equations need to be considered, whose unknowns are non-negative functions $\{f_s(v, t)\}_{s=1}^2$ which represent the density of the opinion $v \in I$, relative to agents of class s, at time $t \geq 0$. The temporal evolution of each distribution function can be described, in spatially homogeneous conditions, as

$$\frac{\partial f_s}{\partial t}(v, t) = \mathcal{I}_s \qquad s \in \{1, 2\} \tag{6}$$

where $\mathcal{I}_s$ is the collisional operator relative to the class s and it is written as

$$\mathcal{I}_s = \sum_{r=1}^{2} \mathcal{Q}_{sr}(f_s, f_r) \qquad s \in \{1, 2\}. \tag{7}$$

From (7) it is evident that the collisional operator relative to each class of agents depends on the distribution functions $\{f_s\}_{s=1}^2$ of all species.

In order to obtain analytic results, the explicit expression of the collisional operator is needed. To simplify notation, in the derivation of the explicit expression of the collisional operator we neglect the dependence of the distribution functions $\{f_s\}_{s=1}^2$ on time t, since all involved integrals are related to the opinion variable. Let us denote as

$$W(v, w, v^*, w^*)\mathrm{d}v^*\mathrm{d}w^* \tag{8}$$

the probability that after the binary interaction of two agents with opinion values v and w, the opinions of the two agents become v^* and w^*, respectively. Hence, the loss of agents of class s in v and, simultaneously, of agents of class r in w can be denoted as

$$\mathcal{Q}_{sr}{}^{-}(f_s, f_r) = W(v, w, v^*, w^*) f_s(v) f_r(w)\mathrm{d}v\mathrm{d}w\mathrm{d}v^*\mathrm{d}w^*.$$

Analogously, the gain of agents of class s in v and, simultaneously, of agents of class r in w, is given by

$$\mathcal{Q}_{sr}{}^{+}(f_s,f_r)=W(v_*,w_*,v,w)f_s(v_*)f_r(w_*)\mathrm{d}v_*\mathrm{d}w_*\mathrm{d}v\mathrm{d}w$$

where v_* and w_* are the pre-interaction opinions of agent of class s and r, respectively, which lead to v and w as opinions of the two agents after the interaction [14].

According to kinetic theory of gas mixtures, the collisional operator $\mathcal{Q}_{sr}$ relative to classes s and r can be written as [15]

$$\begin{aligned}\mathcal{Q}_{sr}(v)=&\int_{I^3}W(v_*,w_*,v,w)f_s(v_*)f_r(w_*)\mathrm{d}v_*\mathrm{d}w_*\mathrm{d}w\\&-\int_{I^3}W(v,w,v^*,w^*)f_s(v)f_r(w)\mathrm{d}w\mathrm{d}v^*\mathrm{d}w^*\end{aligned}\tag{9}$$

where the two integrals are obtained by integrating $\mathcal{Q}_{sr}{}^{-}(f_s,f_r)$ and $\mathcal{Q}_{sr}{}^{+}(f_s,f_r)$ with respect to all the variables except v, and they represents the gain and the loss of agents with opinion in $(v,v+\mathrm{d}v)$, respectively.

Let us now consider the weak form of the Boltzmann equation, which is obtained by multiplying (6) by a test function $\phi(v)$, namely a smooth function with compact support, and integrating the result with respect to v [16]. The weak form of the Boltzmann equation is then given by

$$\int_I\frac{\partial f_s}{\partial t}\phi(v)\mathrm{d}v=\sum_{r=1}^{2}\int_I\mathcal{Q}_{sr}(f_s,f_r)\phi(v)\mathrm{d}v\tag{10}$$

where, according to (9), the integral in the sum on the right hand side can be written as

$$\begin{aligned}&\int_{I^4}W(v_*,w_*,v,w)f_s(v_*)f_r(w_*)\phi(v)\mathrm{d}v_*\mathrm{d}w_*\mathrm{d}v\mathrm{d}w\\&-\int_{I^4}W(v,w,v^*,w^*)f_s(v)f_r(w)\phi(v)\mathrm{d}v\mathrm{d}w\mathrm{d}v^*\mathrm{d}w^*\end{aligned}\tag{11}$$

By applying the change of variables

$$(v_*,w_*,v,w)\to(v,w,v^*,w^*)\tag{12}$$

in the first integral in (11) one obtains that the weak form of the collisional operator $\mathcal{I}_s$ in (11) can be written as

$$\sum_{r=1}^{2}\int_{I^4}W(v,w,v^*,w^*)f_s(v)f_r(w)(\phi(v^*)-\phi(v))\mathrm{d}_4\underline{v}\tag{13}$$

where, from now on, $\mathrm{d}_4\underline{v}$ denotes the products on the four differentials $\mathrm{d}v\mathrm{d}w\mathrm{d}v^*\mathrm{d}w^*$. By substituting (13) in (10) the weak form of the Boltzmann equation for each class $s\in\{1,2\}$ can be finally written as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_I f_s(v,t)\phi(v)\mathrm{d}v=\sum_{r=1}^{2}\int_{I^4}W(v,w,v^*,w^*)f_s(v)\cdot\\ f_r(w)(\phi(v^*)-\phi(v))\mathrm{d}_4\underline{v}\qquad\forall\, s\in\{1,2\}\end{aligned}\tag{14}$$

where on the left hand side we used the fact that for every test function (see [16])

$$\int_I\frac{\partial f_s}{\partial t}\phi(v)\mathrm{d}v=\frac{\mathrm{d}}{\mathrm{d}t}\int_I f_s(v,t)\phi(v)\mathrm{d}v\qquad\forall\, s\in\{1,2\}.\tag{15}$$

III. Analytic Study of Macroscopic Properties

From standard kinetic theory, we can describe the temporal evolution of the distribution function $f_s(v,t)$ according to the spatially homogeneous Boltzmann equation which, in case of a gas mixture, corresponds to (6), where the right hand side represents the collisional operator $\mathcal{I}_s$ relative to the class s. We now show how the Boltzmann equation can be used to derive macroscopic properties of the considered multi-agent system. The number of agents of class s at time t can be expressed as

$$\int_I f_s(v,t)\mathrm{d}v=n_s(t)\qquad s\in\{1,2\}.\tag{16}$$

Similarly, the average opinion of agents of class s at time t can be defined as

$$u_s(t)=\frac{1}{n_s(t)}\int_I f_s(v,t)v\mathrm{d}v\qquad s\in\{1,2\}.\tag{17}$$

Observe that the global (i.e., referred to all the agents) average opinion is then defined as the sum of the average opinions of each class weighed by the number of agents of the corresponding class and divided by n, namely

$$u(t)=\frac{1}{n}\left(n_1(t)u_1(t)+n_2(t)u_2(t)\right).\tag{18}$$

Such definitions are related to two simple test functions $\phi(v)$ in (14). More precisely, setting $\phi(v)=1$ in (14) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I f_s(v,t)\mathrm{d}v=0\qquad s\in\{1,2\}\tag{19}$$

where the 0 on the right hand side is due to the fact that, since $\phi(v)$ is a constant function, the difference $\phi(v^*)-\phi(v)$ inside the integral is 0. Since, from (16), the integral on the left hand side of (19) represents $n_s(t)$, equation (19) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}n_s(t)=0\qquad s\in\{1,2\}\tag{20}$$

so that the number of individuals of each class is conserved. Observe that equation (20) also implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}n(t)=\frac{\mathrm{d}}{\mathrm{d}t}(n_1(t)+n_2(t))=0\tag{21}$$

so that, as expected, that the total number of agents is constant. For these reasons, in the rest of this paper we omit the dependence of n and $\{n_s\}_{s=1}^{2}$ on t. The conservation of the number of agents is a realistic property of the model.

Let us now consider the test function $\phi(v)=v$ in order to investigate the temporal evolution of the average opinion. Setting $\phi(v)=v$ in (14) and using (17) we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_I f_s(v,t)v\mathrm{d}v\\&=\sum_{r=1}^{2}\int_{I^4}W(v,w,v^*,w^*)f_s(v)f_r(w)(v^*-v)\mathrm{d}_4\underline{v}.\end{aligned}\tag{22}$$

Since, from (1), the difference v^*-v can be expressed as $-\gamma_{sr}(v-w)$, the integral on the right hand side of equation (22) che be written as

$$\sum_{r=1}^{2}\gamma_{sr}\int_{I^2}\beta(v,w)f_s(v,t)f_r(w,t)(w-v)\mathrm{d}v\mathrm{d}w\tag{23}$$

where

$$\beta(v,w) = \iint_{I^2} W(v,w,v^*,w^*)\mathrm{d}v^*\mathrm{d}w^* \tag{24}$$

represents the probability of interaction between an agent with opinion v and an agent with opinion w. Using this notation, the weak form of the collisional operator with $\phi(v)=v$ is

$$\sum_{r=1}^{2}\gamma_{sr}\int_{I^2}\beta(v,w)f_s(v,t)f_r(w,t)(w-v)\mathrm{d}v\mathrm{d}w. \tag{25}$$

We now assume that β does not depend on v and w, namely that the probability of interactions between two agents does not depend on their current opinion. Inserting (16) and (17) into (25) and dividing both sides by n_s, the weak form of the Boltzmann equation relative to $\phi(v)=v$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}u_s(t) = \beta\sum_{r=1}^{2}\gamma_{sr}n_r\left(u_r(t)-u_s(t)\right) \qquad s\in\{1,2\}. \tag{26}$$

The 2 equations in (26) represents a homogeneous system of linear differential equations of first order which can be solved analytically. As a matter of fact, let us introduce, for the sake of simplicity, the two parameters

$$a_1 = \beta\gamma_{12}n_2 \qquad a_2 = \beta\gamma_{21}n_1. \tag{27}$$

The two equations in (26) can then be written explicitly as

$$\begin{cases}\dot{u}_1(t) = -a_1(u_1(t)-u_2(t))\\ \dot{u}_2(t) = a_2(u_1(t)-u_2(t)).\end{cases} \tag{28}$$

The solution of the system (28) can be found simply by subtracting the second equation from the first one and, defining $x(t)=u_1(t)-u_2(t)$, we find that

$$\dot{x}(t) = -(a_1+a_2)x(t) \tag{29}$$

whose solution is

$$x(t) = C\mathrm{e}^{-(a_1+a_2)t} \tag{30}$$

and C is an arbitrary constant. Equation (30) implies that

$$u_1(t) = u_2(t) + C\mathrm{e}^{-(a_1+a_2)t}. \tag{31}$$

By substituting (31) in the second equation of (28) one finds

$$\dot{u}_2(t) = Ca_2\mathrm{e}^{-(a_1+a_2)t} \tag{32}$$

where the only unknown is $u_2(t)$ which turns out to be

$$u_2(t) = -C\frac{a_2}{a_1+a_2}\mathrm{e}^{-(a_1+a_2)t} + K. \tag{33}$$

Substituting this result in (31) one finds that the explicit expression of $u_1(t)$ is

$$u_1(t) = C\frac{a_1}{a_1+a_2}\mathrm{e}^{-(a_1+a_2)t} + K. \tag{34}$$

The two constants C and K can be found by imposing that the solutions satisfy the initial conditions, namely

$$\begin{cases}u_1(0) = C\dfrac{a_1}{a_1+a_2} + K\\ u_2(0) = -C\dfrac{a_2}{a_1+a_2} + K\end{cases} \tag{35}$$

where $\{u_j(0)\}_{j=1}^{2}$ are the initial average values of the opinions of the two classes of agents. By subtracting the second equation from the first one, it can be easily shown that

$$C = u_1(0) - u_2(0) \tag{36}$$

and substituting this results in the first equation of (35) gives

$$K = u_1(0)\frac{a_2}{a_1+a_2} + u_2(0)\frac{a_1}{a_1+a_2}. \tag{37}$$

Finally, the solution of (28) obtained by taking into account the initial conditions are

$$\begin{cases}u_1(t) = C\dfrac{a_1}{a_1+a_2}\mathrm{e}^{-(a_1+a_2)t} + K\\ u_2(t) = -C\dfrac{a_2}{a_1+a_2}\mathrm{e}^{-(a_1+a_2)t} + K\end{cases} \tag{38}$$

where C and K are defined in (36) and (37), respectively. From (38) it is clear that the following limits hold

$$\lim_{t\to+\infty}u_1(t) = \lim_{t\to+\infty}u_2(t) = K. \tag{39}$$

Observe that, according to (37) and (27), the value of the limit K depends on the average intial opinions $\{u_s(0)\}_{s=1}^{2}$, on the number of agents $\{n_s\}_{s=1}^{2}$ in each class, and on γ_{12} and γ_{21}.

We are now interested in studying the convergence time. In particular, since $|u_s(t)-K|$ represents the distance between the average opinion of the classes s at time t and its limit for $t\to+\infty$, we consider the following inequalities

$$|u_1(t)-K| \le \varepsilon \qquad |u_2(t)-K| \le \varepsilon. \tag{40}$$

From (38) the first inequality in (40) is equivalent to

$$\mathrm{e}^{-(a_1+a_2)t} \le \varepsilon\frac{a_1+a_2}{|C|a_1}. \tag{41}$$

From (41) it can be concluded that

$$|u_1(t)-K| \le \varepsilon \iff t \ge t_1 = \frac{1}{a_1+a_2}\log\left(\frac{a_1}{a_1+a_2}\frac{|C|}{\varepsilon}\right).$$

Analogous elaborations show that

$$|u_2(t)-K| \le \varepsilon \iff t \ge t_2 = \frac{1}{a_1+a_2}\log\left(\frac{a_2}{a_1+a_2}\frac{|C|}{\varepsilon}\right).$$

Finally, one can evaluate the minimum time necessary to ensure that the solution $u_1(t)$ differs from $u_2(t)$ for no more than ε. From (38) one obtains

$$|u_1(t)-u_2(t)| = |C|\mathrm{e}^{-(a_1+a_2)t} \tag{42}$$

so that

$$|u_1(t)-u_2(t)| \le \varepsilon \iff t \ge t_{\min} \tag{43}$$

with

$$t_{\min} = \frac{1}{a_1+a_2}\log\left(\frac{|C|}{\varepsilon}\right). \tag{44}$$

Observe that $t_{\min}$ is the minimum value of the time t which guarantees that the average opinions of the two classes of agents differ less than ε. The condition (43) is only relative to the average opinions and does not imply consensus.

TABLE I. THE CONSIDERED VALUES OF THE PARAMETERS FOR THE TWO CLASSES OF AGENTS: NUMBER OF AGENTS (FIRST AND SECOND COLUMN); INITIAL DISTRIBUTIONS OF THE OPINION (THIRD AND FOURTH COLUMN); PARAMETERS γ_{sr} (FIFTH AND SIXTH COLUMN).

n_1	n_2	$f_1(v,0)$	$f_2(v,0)$	γ_{12}	γ_{21}
500	500	$\mathcal{U}((-1;1/3))$	$\mathcal{U}((-1/3;1))$	5/100	10/100
750	250	$\mathcal{U}((-1;1/3))$	$\mathcal{U}((-1/3;1))$	5/100	10/100
900	100	$\mathcal{U}((-1;1))$	$\mathcal{U}((3/4;1))$	10/100	1/100

IV. VERIFICATION OF RESULTS BY SIMULATION

In this section, we show simulation results concerning the opinion dynamics according to the framework proposed in Section II. We remark that such results are obtained by implementing the microscopic equations in (1), thus neglecting the analytic framework relative to the Boltzmann equation. From now on we denote as $\{\tilde{u}_s(t)\}_{s=1}^2$ the values of the average opinions of the class s found by simulation while $\{u_s(t)\}_{s=1}^2$ represent the analytic solutions in (38). We consider a system composed of $n = 10^3$ agents. Table I shows the values of the parameters relative to the two classes of agents which are considered to derive analytic and simulation results in this section. In particular, different values of the parameters are considered for: (*i*) the number of agents $\{n_s\}_{s=1}^2$; (*ii*) the initial distribution of opinion; and (*iii*) the values of $\{\gamma_{sr}\}_{s,r=1}^2$.

First, we consider the parameters shown in the first row of Table I. In this case, $n_1 = n_2 = 500$, namely the two classes of agents have the same number of agents. The initial opinions of the agents of class 1 are uniformly distributed in the interval $(-1;1/3)$, so that the initial average opinion of the agents of class 1 is $u_1(0) = -1/3$. The initial opinions of the agents of class 2, instead, are uniformly distributed in the interval $(-1/3;1)$, and, therefore, their initial average opinion is $u_2(0) = 1/3$. The two classes of agents are not only distinguished by their initial opinion distribution but they are also characterized by different propensity at changing their opinions when interacting with other agents. More precisely, the value of γ_{12} is $5/100$ while the value of γ_{21} is $10/100$. Since $\gamma_{21} = 2\gamma_{12}$, the agents of class 2 are more inclined to change their opinion than those of class 1. Fig. 1 shows the average opinion $u_1(t)$ of the agents of class 1 (blue line) and the average opinion $u_2(t)$ of the agents of class 2 (red line). As expected from (37), $u_1(t)$ and $u_2(t)$ converge to the same value, which, according to this choice of parameters, corresponds to $K = -1/9$. Fig. 1 also shows the values of $\{\tilde{u}_s(t)\}_{s=1}^2$ obtained by simulation. More precisely, the dashed cyan line refers to $\tilde{u}_1(t)$ while the dashed magenta line refers to $\tilde{u}_2(t)$. It can be observed that analytic results are in agreement with those obtained by simulating pairwise interactions according to (1). In Fig. 1, the value of the average opinion $u(t)$ defined in (18) is also shown (dash-dotted black line). As expected from Section III, $u(t)$ also converges to K.

Fig. 2 shows the distribution $f_1(v,t)$ (blue lines) and $f_2(v,t)$ (red lines) of the opinions of the two classes of agents obtained by simulating the multi-agent system with the parameters shown in the first line of Table I. More precisely: Fig. 2 (a) shows the distributions $f_s(v,t)$ after 10^4 interactions; Fig. 2 (b) shows the distributions $f_s(v,t)$ after $2\cdot10^4$ interactions; Fig. 2 (c) shows the distributions $f_s(v,t)$ after $3\cdot10^4$ interactions; and Fig. 2 (d) shows the distributions $f_s(v,t)$ after 10^5 interactions. From Fig. 2 it can be observed that not only the average opinions $u_s(t)$ converge to the same value K, but also that, as discussed in previous sections, consensus among agents is reached, since the opinions of each agents tend to the same value.

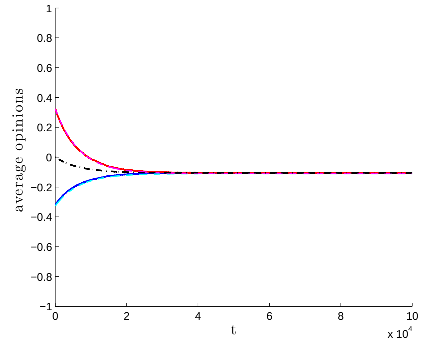


Fig. 1. The average opinions $u_1(t)$ (blue line) and $u_2(t)$ (red line) derived analytically with the parameters in the first row of Table I are shown. The corresponding average opinion $u(t)$ is also shown (dash-dotted black line). The values of $\tilde{u}_1(t)$ (dashed cyan line) and $\tilde{u}_2(t)$ (dashed magenta line) obtained by simulation are shown.

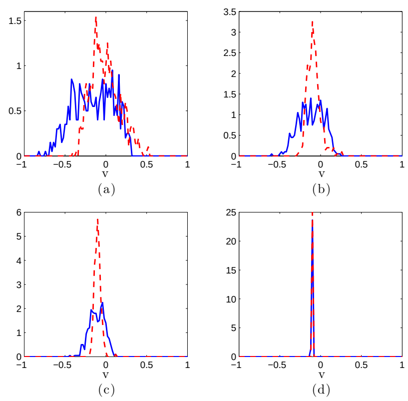
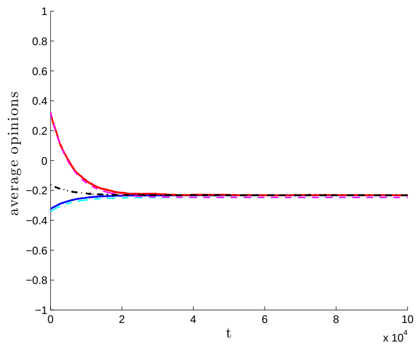


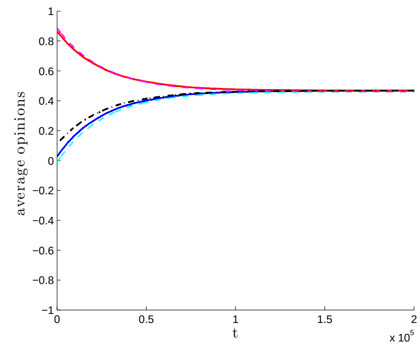
Fig. 2. The opinion distributions $f_1(v,t)$ (solid blue line) and $f_2(v,t)$ (dashed red line) relative to the parameters in the first row of Table I are shown: (a) after 10^4 interactions; (b) after $2\cdot10^4$ interactions; (c) after $3\cdot10^4$ interactions; (d) after 10^5 interactions.

We now consider the parameters shown in the second row of Table I. In this case, the two classes of agents differ not only because of their initial distribution of opinions and their values of γ_{sr} (which are equal to those previously considered), but also because of the number of agents. More precisely, agents of class 1 represent 75% of the population. Fig. 3 (a) shows the average opinion $u_1(t)$ of the agents of class 1 (blue line) and the average opinion $u_2(t)$ of the agents of class 2 (red line). As expected from (37), the values of $u_1(t)$ and $u_2(t)$ converge to the same value, which, with these new values of $\{n_s\}_{s=1}^2$, corresponds to $K \simeq -0.24$. The values of $\tilde{u}_1(t)$ (dashed cyan line) and $\tilde{u}_2(t)$ (dashed magenta line) obtained by simulation are also shown in Fig. 3 and they are in agreement with those obtained analytically. Fig. 3 (a) also shows the value of the average opinion $u(t)$ (dash-dotted black line) defined in (18), which converges to the same value K.

Finally, we consider the parameters shown in the third row of Table I. In this case, $n_1 = 900$ and $n_2 = 100$, i.e., agents of class 2 represent only 10% of the entire population. Under this assumption we consider that the initial opinions of the



(a)



(b)

Fig. 3. The average opinions $u_1(t)$ (blue line) and $u_2(t)$ (red line) derived analytically are compared to $\tilde{u}_1(t)$ (dashed cyan line) and $\tilde{u}_2(t)$ (dashed magenta line) obtained by simulation when considering the parameters: (a) in the second row of Table I and (b) in the third row of Table I. The corresponding average opinion $u(t)$ is also shown (dash-dotted black line).

agents of class 1 are uniformly distributed in the interval I (so that $u_1(0) = 0$) and the initial opinions of the agents of class 2 are uniformly distributed in the interval $(3/4; 1)$ (so that $u_2(0) = 7/8$). This choice corresponds to considering agents of class 2 as extremists, since their opinions are very close to one of the extremes of the interval I. In agreement with the idea that extremal opinions are typically more difficult to be changed, we assume that the value of γ_{21} is smaller than γ_{12}. More precisely, we consider $\gamma_{12} = 1/10$ and $\gamma_{21} = 1/100$, so that γ_{21} is equal to a tenth of γ_{12}. According to the choice $\gamma_{21} = 1/100$, agents of class 2 are skeptical.

Fig. 3 (b) shows the average opinion $u_1(t)$ of the agents of class 1 (blue line) and the average opinion $u_2(t)$ of the agents of class 2 (red line) as functions of time t. As in the previous cases, $u_1(t)$ and $u_2(t)$ converge to the same value, which, according to this choice of parameters and (37), corresponds to $K \simeq 0.46$. Fig. 3 (b) also shows the values of $\tilde{u}_1(t)$ (dashed cyan line) and $\tilde{u}_2(t)$ (dashed magenta line) obtained by simulation. Once again, analytic results obtained according to the kinetic approach are in agreement with those obtained by simulations. For the sake of completeness, Fig. 3 (b) also shows the value of the average opinion $u(t)$ (dash-dotted green line) defined in (18). As expected, $u(t)$ also converges to K. A wide variety of choices for the parameters of the model could be taken and the results shown here are only illustrative of some particular configurations. The agreement between analytic and simulation results indicates that the framework based on kinetic theory is consistent and, therefore, it can be properly used to analytically study opinion dynamics.

V. Conclusions

In this paper, we study analytically a model for opinion dynamics based on kinetic theory. We start from the description of the effects of microscopic interactions among agents, which are assumed to be binary, and we describe macroscopic properties related to opinion dynamics in the considered multi-agent system, using proper balance equations. More precisely, we take inspiration from kinetic theory of gas mixtures, which allows describing the behavior of gases composed of different kinds of molecules. Similarly, we aim at describing a multi-agent system composed of different classes of agents. The considered different classes of agents have different characteristics, namely: (*i*) cardinality, (*ii*) initial average opinions, and (*iii*) propensity to change opinions.

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