A Domain View of Timed Behaviors ⋆

Roman Dubtsov[1], Elena Oshevskaya[2], and Irina Virbitskaite[2]

[1] Institute of Informatics System SB RAS,
6, Acad. Lavrentiev av., 630090, Novosibirsk, Russia;
[2] Institute of Mathematics SB RAS,
4, Acad. Koptyug av., 630090, Novosibirsk, Russia;
dubtsov,eso,virb@iis.nsk.su

Abstract. The intention of this paper is to introduce a timed extension of transition systems with independence, and to study its categorical interrelations with other timed "true-concurrent" models. In particular, we show the existence of a chain of coreflections leading from a category of the model of timed transition systems with independence to a category of a specially defined model of marked Scott domains. As an intermediate semantics we use a model of timed event structures, able to properly capture causality, conflict, and concurrency among events which arise in the presence of time delays of the events.

1 Introduction

The behaviour of concurrent systems is often specified in terms of states and transitions between states, the labels on the transitions represent the observable part of system's behaviour. The simplest formal model of computation able to express naturally this idea is that of labelled transition systems. However, they are a representative of the interleaving approach to concurrency and hence do not allow one to draw a natural distinction between interleaved and concurrent executions of system's actions. Two most popular "true concurrent" extensions of transition systems, aiming to overcome limitations of the interleaving approach, are asynchronous transition systems, introduced independently by Bednarczyk [1] and Shields [2], and transitions systems with independence, proposed by Winskel and Nielsen [3].

Category theory [4] has been successfully exploited to structure the tangled world of models for concurrency. Within this framework, objects of categories represent processes and morphisms correspond to behavioural relations between the processes, i.e. to simulations. The category-theoretic approach allows for natural formalization of the fact that one model is more expressive than another in terms of an "embedding", most often taking the form of a coreflection, i.e. an adjunction in which the unit is an isomorphism. For example, Hildenbrandt and

⋆ The second author is supported in part by the RFBR (grant 12-01-00873-a), by the President Program "Leading Scientific Schools" (grant NSh-7256.2010.1), and by the Federal Program "Research and educational personnel for innovative Russia" (grant 8206).

Sassone [5] have constructed a full subcategory of a category of asynchronous transition systems and have shown the existence of a coreflection between the subcategory and a category of transition systems with independence. In their next paper [6], the authors have enriched the model of transition systems with independence by adding multi-arcs and have yielded a precise characterization of the model in terms of (event-maximal, diamond-extensional) labeled asynchronous transition systems, by constructing functors between categories of the models.

It is generally acknowledged that time plays an important role in many concurrent and distributed systems. This has motivated the lifting of the theory of untimed systems to the real-time setting. Timed transition system like models have been studied thoroughly within the two last decades (see [7,8] among others), while timed "true concurrent" extensions have hitherto received scant attention.

The aim of this paper is to introduce a timed extension of transition systems with independence, and to study its categorical interrelations with other timed "true-concurrent" models. In particular, we show the existence of a chain of coreflections leading from a category of the model of timed transition systems with independence to a category of a specially defined model of marked Scott domains. As an intermediate semantics we use a model of timed event structures, able to properly capture causality, concurrency, and conflict among events which arise in the presence of time delays of the events.

The paper is organized as follows. In Section 2, the notions and notations concerning the structure and behaviour of timed transition systems with independence are described. Also, an unfolding of timed transition systems with independence is constructed, and it is shown that together with the inclusion functor the unfolding functor defines a coreflection. Section 3 establishes the interrelations in terms of the existence of a coreflection between timed occurrence transition systems with independence and timed event structures. In Section 4, using the equivalence of the categories of timed event structures and marked Scott domains, stated in [9], functors between the categories of timed transition systems with independence and marked Scott domains are constructed to constitute a coreflection. Section 5 provides a direct translation from timed transition systems with independence to marked Scott domains, established in the categorical setting. In section 6, we conclude with a short summary of the discovered relationships.

2 Timed Transition Systems with Independence

In this section, we first describe the basic notions and notations concerning the structure and behaviour of timed transition systems with independence.

We start with untimed case. A *transition system with independence* is a tuple $TI = (S, s^I, L, Tran, I)$, where S is a countable set of *states*, $s^I \in S$ is the *initial state*, L is a countable set of *labels*, $Tran \subseteq S \times L \times S$ is the *transition* relation, and $I \subseteq Tran \times Tran$ is the irreflexive, symmetric *independence relation*, such that, using $\prec$ to denote the following relation on transitions $(s, a, s') \prec (s'', a, u) \iff \exists (s, b, s''), (s', b, u) \in Tran$ s.t. $(s, a, s')\ I\ (s, b, s'') \wedge (s, a, s')\ I\ (s', b, u) \wedge (s, b, s'')\ I\ (s'', a, u)$, and $\sim$ for the least equivalence relation containing $\prec$, we have:

1. $(s, a, s') \sim (s, a, s'') \Rightarrow s = s''$,
2. $(s, a, s')\ I\ (s, b, s'') \Rightarrow \exists (s', b, u), (s'', a, u) \in Tran\ .\ (s, a, s')\ I\ (s', b, u) \wedge (s, b, s'')\ I\ (s'', a, u)$,
3. $(s, a, s')\ I\ (s', b, u) \Rightarrow \exists (s, b, s''), (s'', a, u) \in Tran\ .\ (s, a, s')\ I\ (s, b, s'') \wedge (s, b, s'')\ I\ (s'', a, u)$,
4. $(s, a, s') \sim (s'', a, u)\ I\ (w, b, w') \Rightarrow (s, a, s')\ I\ (w, b, w')$.

Let $\mathrm{Diam}_{a,b}(s, s', s'', u) \iff \exists (s, a, s'), (s, b, s''), (s', b, u), (s'', a, u) \in Tran\ .\ (s, a, s')\ I\ (s, b, s'') \wedge (s, a, s')\ I\ (s', b, u) \wedge (s, b, s'')\ I\ (s'', a, u)$. We say that the transitions above form an *independence diamond*, and denote the $\sim$-equivalence class of a transition $t \in Tran$ as $[t]$.

A transition system with independence functions by executing transitions from one state to another. A possibly infinite sequence $\pi = t_0\ t_1\ \ldots$ with $t_i = (s_i, a_i, s_{i+1}) \in Tran$ $(i \geq 0)$ is called a *path*. The starting state of π is denoted as $\mathrm{dom}(\pi)$, and the ending state as $\mathrm{cod}(\pi)$ if π is a finite path. A *computation* is a path π such that $\mathrm{dom}(\pi) = s^I$. Let $\mathrm{Comp}(TI)$ ($\mathrm{Comp}^0(TI)$) be the set of all (finite) computations of TI. A transition t is said to be *reachable*, if there exists a computation $\pi \in \mathrm{Comp}^0(TI)$ such that t appears in π. From now on, we consider only those transition systems with independence in which all transitions are reachable. Let $\simeq \subseteq \mathrm{Comp}(TI) \times \mathrm{Comp}(TI)$ be the least equivalence relation such that $\pi_s(s, a, s')(s', b, u)\pi_v \simeq \pi_s(s, b, s'')(s'', a, u)\pi_v \iff \mathrm{Diam}_{a,b}(s, s', s'', u)$, and let $[\pi]$ stand for the $\simeq$-equivalence class of a computation π.

We now incorporate time into the model of transition systems with independence. By analogy with the paper [8], we assume a global, fictitious clock, whose actions advance time by nonuniform amounts and whose value is set to zero at the beginning of system's functioning. All transitions are associated with timing constraints represented as minimal and maximal time delays, and happen "instantaneously", while timing constraints restrict the times at which transitions may be executed. Unlike the paper [8], in our timed model the time domain is changed to the integers, and the maximal delays associated with transitions are always equal to ∞, therefore they are not specified explicitly.

Let $\mathbb{N}$ be the set of non-negative integers.

Definition 1. *A* timed transition system with independence *is a tuple* $TTI = (S, s^I, L, Tran, I, \delta)$*, where* $[\![TTI]\!] = (S, s^I, L, Tran, I)$ *is the underlying transition system with independence, and* $\delta : Tran \to \mathbb{N}$ *is the* delay function *such that* $\delta(t) = \delta(t')$ *for any* $t, t' \in Tran$ *such that* $t \sim t'$.

A *timed computation* of a timed transition system with independence $TTI = (S, s^I, L, Tran, I, \delta)$ is a pair $\Pi = (\pi, \tau) \in (\mathrm{Comp}((S, s^I, L, Tran, I)) \times (\mathbb{N} \cup \{\infty\}))$ with $\tau \geq \delta(\pi) = sup\{\delta(t) \mid t \in \pi\}$. Define $\mathrm{dom}(\Pi) = \mathrm{dom}(\pi)$ and $\mathrm{cod}(\Pi) = \mathrm{cod}(\pi)$. We denote the set of all (finite) timed computations of TTI as

TComp(TTI) (TComp^0(TTI)), and write $\Pi \simeq_\tau \Pi'$ iff $\pi \simeq \pi'$ and $\tau = \tau'$. It is easy to see that $\simeq_\tau$ is an equivalence relation; the $\simeq_\tau$-equivalence class of a timed computation Π is denoted as $[\Pi]_\tau$. Let TComp$_{\simeq_\tau}$(TTI) (TComp$^0_{\simeq_\tau}$(TTI)) be the sets of $\simeq_\tau$-equivalence classes of all (finite) timed computations of TTI.

For timed transition systems with independence $TTI = (S, s^I, L, Tran, I, \delta)$ and $TTI' = (S', s'^I, L', Tran', I', \delta')$, a *morphism* $h : TTI \to TTI'$ is a pair of mappings $h = (\sigma : S \to S', \lambda : L \to^* L')$[3] such that:

1. $\sigma(s^I) = s'^I$,
2. $(s, a, s') \in Tran \;\Rightarrow\; (\sigma(s), \alpha(a), \sigma(s') \in Tran'$ if $a \in \operatorname{dom}\lambda$, and $\sigma(s) = \sigma(s')$, otherwise,
3. $(s, a, s')I(\bar{s}, \bar{a}, \bar{s}')$ and $a, \bar{a} \in \operatorname{dom}\lambda \;\Rightarrow\; (\sigma(s), \alpha(a), \sigma(s')I'(\sigma(\bar{s}), \alpha(\bar{a}), \sigma(\bar{s}')$,
4. $\delta'((\sigma(s), \alpha(a), \sigma(s'))) \leq \delta((s, a, s'))$.

Timed transition systems with independence and morphisms between them form a category **TTSI** with unit morphisms $\mathbf{1}_{TTI} = (\mathbf{1}_S, \mathbf{1}_L) : TTI \to TTI$ for any $TTI = (S, s^I, L, Tran, I, \delta)$, and with composition defined in a componentwise manner.

We next aim at unfolding of timed transition systems with independence. To that end, we first define a subclass of timed transition systems with independence that serves as a target of unfolding. After that, we construct an unfolding mapping and show that together with the inclusion functor the unfolding functor defines a coreflection.

Definition 2. *A* timed occurrence transition system with independence $ToTI = (S, s_0, L, Tran, I, \delta)$ *is an acyclic timed transition system with independence such that* $(s'', a, u) \neq (s', b, u) \in Tran \Rightarrow \exists s \in S$ *s.t.* $\mathrm{Diam}_{a,b}(s, s', s'', u)$*, for all* $(s'', a, u), (s', b, u) \in Tran$.

Let **ToTSI** be the full subcategory of the category **TTSI**.

Define an unfolding mapping *ttsi.totsi* : **TTSI** $\to$ **ToTSI** as follows. For a timed transition system with independence $TTI = (S, s^I, L, Tran, I, \delta)$, specify *ttsi.totsi*($TTI$) as $(S_{\simeq_\tau}, [(s^I, 0)]_\tau, L, Tran_{\simeq_\tau}, I_{\simeq_\tau}, \delta_{\simeq_\tau})$, where

- $S_{\simeq_\tau} = \{[\Pi = (\pi, \delta(\pi))]_\tau \in \mathrm{TComp}^0_{\simeq_\tau}(TTI)\}$,
- $([\Pi = (\pi, \delta(\pi))]_\tau, a, [\Pi' = (\pi', \delta(\pi'))]_\tau) \in Tran_{\simeq_\tau} \iff \exists t_{\pi,\pi'} = (s, a, s') \in Tran$. $\Pi' \simeq_\tau (\pi t_{\pi,\pi'}, \max\{\delta(\pi), \delta(\pi')\})$,
- $([\Pi]_\tau, a, [\Pi']_\tau) I_{\simeq_\tau} ([\bar{\Pi}]_\tau, b, [\bar{\Pi}']_\tau) \iff t_{\pi,\pi'} I t_{\bar{\pi},\bar{\pi}'}$,
- $\delta_{\simeq_\tau}([\Pi]_\tau, a, [\Pi']_\tau) = \delta(t_{\pi,\pi'})$.

Lemma 1. *Given a timed transition system with independence* TTI*, ttsi.totsi*(TTI) *is a timed occurrence transition system with independence.*

[3] A partial mapping from a set A into a set B is denoted as $f : A \to^* B$. Let $\operatorname{dom} f = \{a \in A \mid f(a)$ is defined$\}$. For a subset $A' \subseteq A$, define $fA' = \{f(a') \mid a' \in A' \cap \operatorname{dom} f\}$.

In order to demonstrate that the mapping *ttsi.totsi* is adjoint to the inclusion functor **ToTSI** $\hookrightarrow$ **TTSI**, we define a mapping and prove that it is the unit of this adjunction. For a transition system with independence TTI, let $\varepsilon_{TTI} = (\sigma_\varepsilon, 1_L) : ttsi.totsi(TTI) \to TTI$, where $\sigma_\varepsilon([\Pi]_\tau) = \mathrm{cod}(\Pi)$ for all $[\Pi]_\tau \in S_{\simeq_\tau}$. It is easy to see that ε_{TTI} is a morphism of **TTSI**.

Lemma 2 (ε_{TTI} is couniversal). *For any object TTI of* **TTSI**, *any object $ToTI$ of* **ToTSI** *and any morphism $h : ToTI \to TTI$ of* **TTSI**, *there exists a unique morphism $h' : ToTI \to ttsi.totsi(TTI)$ of* **ToTSI** *such that $h = \varepsilon_{TTI} \circ h'$.*

The next theorem presents a categorical characterization of the unfolding.

Theorem 1 ($\hookrightarrow \dashv$ *ttsi.totsi*). *The unfolding mapping ttsi.totsi extends to a functor from* **TTSI** $\to$ **ToTSI** *which is right adjoint to the functor* $\hookrightarrow$: **ToTSI** $\to$ **TTSI**. *Moreover, this adjunction is a coreflection.*

3 Timed Event Structures

In this section we relate timed occurrence transition systems with independence and timed event structures, establishing the close relationships between categories of the models.

We start with the definition of an untimed variant of event structures. An *event structure* is a triple $\mathcal{E} = (E, \leq, \#)$, where E is a countable set of *events*; $\leq \subseteq E \times E$ is a partial order (*the causality relation*) such that $\downarrow e = \{e' \in E \mid e' \leq e\}$ is a finite set for each $e \in E$, $\# \subseteq E \times E$ is the symmetric irreflexive *conflict relation* such that $e \# e' \leq e'' \Rightarrow e \# e''$. A set of events $C \subseteq E$ is said to be a *configuration* of an event structure $\mathcal{E}$ if $\forall e \in C \,.\, \downarrow e \subseteq C$, and $\forall e, e' \in C \,.\, \neg(e \# e')$. We say that events $e, e' \in E$ are *concurrent* and write $e \smile e'$ if $\neg(e \leq e' \vee e' \leq e' \vee e \# e')$. Introduce the concept of a *reflexive conflict* as follows: $e \,\text{\textit{w}}\, e' \iff e \# e' \vee e = e'$.

We now recall the definition of timed event structures from [9]. Similarly to the model of timed transition systems with independence, there is a global non-negative integer-valued clock. Each event in the structure is associated with a time delay with respect to the initial time moment; i.e., if an event e is associated with a time delay t, then e may not occur earlier than all the predecessors of the event occur and the clock shows time t. In this case, the event itself occurs instantaneously.

Definition 3. *A* timed event structure *is a tuple $\mathcal{TE} = (E, \leq, \#, \Delta)$, where $(E, \leq, \#)$ is an event structure and $\Delta : E \to \mathbb{N}$ is the* delay function *such that $e' \leq e \Rightarrow \Delta(e') \leq \Delta(e)$.*

A *timed configuration* of $\mathcal{TE}$ is a pair (C, τ), where C is a configuration of $(E, \leq, \#)$ and $\tau \in \mathbb{N} \cup \{\infty\}$ such that $\tau \geq \Delta(C) = sup\{\Delta(e) \mid e \in C\}$. The set of all (finite) timed configurations of a timed event structure $\mathcal{TE}$ is denoted as $\mathrm{TConf}(\mathcal{TE})$ ($\mathrm{TConf}^0(\mathcal{TE})$). We define a transition relation $\longrightarrow$ on the set

$\mathrm{TConf}(\mathcal{TE})$ as follows: $(C,t) \longrightarrow (C',t')$ if $C \subseteq C'$ and $t \leq t'$. Clearly, the relation $\longrightarrow$ specifies a partial order on the set $\mathrm{TConf}(\mathcal{TE})$.

Let $\mathcal{TE} = (E, \leq, \#, \Delta)$ and $\mathcal{TE}' = (E', \leq', \#', \Delta')$ be timed event structures. A partial mapping $\theta : E \rightarrow^* E'$ is a *morphism* if $\downarrow\theta(e) \subseteq \theta \downarrow e$; $\theta(e) \veebar \theta(e') \Rightarrow e \veebar e'$, for all $e, e' \in \mathrm{dom}\,\theta$; $\Delta'(\theta(e)) \leq \Delta(e)$, for all $e \in \mathrm{dom}\,\theta$. Timed event structures with their morphisms define a category **TES** with unit morphisms $\mathbf{1}_{TS} = \mathbf{1}_E : TS \to TS$ for all $TS = (E, \leq, \#, \Delta)$ and the composition being a usual composition of partial functions.

We now establish the relationships between the categories of timed event structures and timed occurrence transition systems with independence. For this purpose, we first define a mapping *tpes.totsi* : **TPES** $\to$ **ToTSI** extending the mapping *pes.otsi* from [3] to the timed case. For a timed event structure $\mathcal{TE} = (E,\leq,\#,\Delta)$, let *tpes.totsi*$(\mathcal{TE})$ be $(S_{\mathcal{TE}}, s^I_{\mathcal{TE}}, L_{\mathcal{TE}}, Tran_{\mathcal{TE}}, I_{\mathcal{TE}}, \delta_{\mathcal{TE}})$, where

- $S_{\mathcal{TE}} = \big\{(C, \Delta(C)) \in \mathrm{TConf}^0(\mathcal{TE})\big\}$;
- $s^I_{\mathcal{TE}} = (\varnothing, 0)$;
- $L_{\mathcal{TE}} = E$;
- $\big((C, \Delta(C)), e, (C', \Delta(C'))\big) \in Tran_{\mathcal{TE}} \iff C' \setminus C = \{e\}$;
- $\big((C, \Delta(C)), e, (C', \Delta(C'))\big) I_{\mathcal{TE}} \big((\bar{C}, \Delta(\bar{C})), \bar{e}, (\bar{C}', \Delta(\bar{C}'))\big) \iff e \smile \bar{e}$;
- $\delta_{\mathcal{TE}}((C, \Delta(C)), e, (C', \Delta(C'))) = \Delta(e)$.

It is easy to see that the above definition is correct, i.e. *tpes.totsi* maps timed event structures to timed occurrence transition systems with independence.

Next, we construct a mapping *totsi.tpes* : **ToTSI** $\to$ **TPES**. For a timed occurrence transition system with independence $ToTI = (S, s^I, L, Tran, I, \delta)$, let *totsi.tpes*$(ToTI)$ be $(Tran_\sim, \leq, \#, \Delta)$, where

- $Tran_\sim = \{[t] \mid t \in Tran\}$,
- $[t] < [t'] \iff$
  $\forall (\pi\bar{t}', \tau) \in \mathrm{TComp}^0(ToTI) \,.\, \bar{t}' \sim t' \Rightarrow (\exists \bar{t} \in \pi \,.\, \bar{t} \sim t)$; $\leq = < \cup =$,
- $[t] \# [t'] \iff$
  $\forall (\pi, \tau) \in \mathrm{TComp}^0(ToTI), \forall \bar{t} \in [t], \forall \bar{t}' \in [t'] \,.\, \bar{t} \in \pi \Rightarrow \bar{t}' \notin \pi$,
- $\Delta([t]) = \max\{\delta(t') \mid [t'] \leq [t]\}$.

On morphisms $h = (\sigma, \lambda) : ToTI \to ToTI'$ in **ToTSI**, the mapping *totsi.tpes* acts as follows: *totsi.tpes*$(h) = \theta$, where $\theta([(s, a, s')]) = [(\sigma(s), \lambda(a), \sigma(s')]$, if $a \in \mathrm{dom}\,\lambda$, and $\theta([(s, a, s')])$ is undefined, otherwise.

Proposition 1. *totsi.tpes* : **ToTSI** $\to$ **TPES** *is a functor.*

Finally, we define the unit of the adjunction. For a timed event structure $\mathcal{TE}$, let $\eta_{\mathcal{TE}} : E_{\mathcal{TE}} \to E_{totsi.tpes \circ tpes.totsi(\mathcal{TE})}$ be a mapping such that $\eta_{\mathcal{TE}}(e) = [(C, \Delta(C)), e, (C \cup \{e\}, \Delta(C \cup \{e\}))]$. It is straightforward to show that $\eta_{\mathcal{TE}}$ is an isomorphism in **TPES**. In order to demonstrate the existence of the adjunction, we need to check that $\eta_{\mathcal{TE}}$ is indeed a unit, i.e. it is universal.

Lemma 3 ($\eta_{\mathcal{TE}}$ is universal).
For any object $\mathcal{TE}$ of **TPES***, any object $ToTI$ of* **ToTSI***, and any morphism $\theta : \mathcal{TE} \to totsi.tpes(ToTI)$ in* **TPES***, there exists a unique morphism $h : tpes.totsi(\mathcal{TE}) \to ToTI$ in* **ToTSI** *such that $\theta = totsi.tpes(h) \circ \eta_{\mathcal{TE}}$.*

The next theorem establishes the existence of a coreflection between the categories of timed event structures and timed occurrence transition systems with independence.

Theorem 2 (*tpes.totsi* $\dashv$ *totsi.tpes*)**.** *The map tpes.totsi can be extended to a functor tpes.totsi* : **TPES** $\to$ **ToTSI***, which is left adjoint to the functor totsi.tpes. Moreover, this adjunction is a coreflection.*

4 Marked Scott Domains

In this section, we extend the established chain of coreflections to marked Scott domains. To that end, we first recall related notions and notations.

Let $(D, \sqsubseteq)$ be a partial order, $d \in D$ and $X \subseteq D$. Then,

- $\uparrow d = \{d' \in D \mid d \sqsubseteq d'\}$ is an *upper cone of element* d, $\downarrow d = \{d' \in D \mid d' \sqsubseteq d\}$ is a *lower cone of element* d,
- X is downward (upward) closed if $\downarrow d \subseteq X$ ($\uparrow d \subseteq X$) for every $d \in X$,
- X is a *compatible set* (denoted as $X\uparrow$), if the following assertion is true: $\exists d \in D \forall x \in X \,.\, x \sqsubseteq d$, i.e., X has an upper bound. If $X = \{x, y\}$, we write $x \uparrow y$ instead of $\{x, y\}\uparrow$. The least upper bound of the set X is denoted as $\bigsqcup X$ (if it exists), and the greatest lower bound is denoted as $\bigsqcap X$ (if it exists). The least upper bound of two elements x and y is denoted as $x \sqcup y$, and the greatest lower bound, as $x \sqcap y$.
- X is a *finitely compatible set* if any finite subset of it $X' \subseteq X$ is compatible.
- X is a *(upper) directed set* if any finite subset of it $X' \subseteq X$ has an upper bound belonging to the set X (thus, X is a finitely compatible and nonempty set).
- $(D, \sqsubseteq)$ is a *directed-complete partial order (dcpo for short)* if every directed subset $X \subseteq D$ has $\bigsqcup X$.
- d is a *finite (compact) element* of a dcpo $(D, \sqsubseteq)$ if, for any directed subset $X \subseteq D$, the following assertion is true: $d \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X \,.\, d \sqsubseteq x$. The set of finite elements is denoted as $C(D)$.
- A dcpo $(D, \sqsubseteq)$ is said to be *algebraic* if, for any $d \in D$, $d = \bigsqcup\{e \sqsubseteq d \mid e \in C(D)\}$. It is said to be ω-algebraic if $C(D)$ is countable.
- $(D, \sqsubseteq)$ is a *consistently complete partial order* (ccpo) if any finitely compatible subset $X \subseteq D$ has $\bigsqcup X$. Clearly, a ccpo has the least element $\perp = \bigsqcup \emptyset$, and is also a dcpo.
- An ω-algebraic ccpo is called a *Scott domain.* A Scott domain $(D, \sqsubseteq)$ is said to be *finitary* if $\downarrow d$ is finite for every $d \in C(D)$.

Describe some properties of Scott domains. An element p of a Scott domain $(D, \sqsubseteq)$ is said to be *prime* if, for any compatible subset $X \subseteq D \,.\, p \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X \,.\, p \sqsubseteq x$. The set of the prime elements is denoted as $P(D)$. A Scott domain $(D, \sqsubseteq)$ is called *prime algebraic* if, for any $d \in D$, $d = \bigsqcup\{p \sqsubseteq d \mid p \in P(D)\}$ and *coherent* if all subsets $X \subseteq D$ satisfying the condition $\forall d', d'' \in X \,.\, d' \uparrow d''$ have $\bigsqcup X$.

Let $(D, \sqsubseteq)$ be a Scott domain and $\prec \, = \, \sqsubset \setminus \sqsubset^2$ be a *covering relation*. For elements $d, d' \in D$ such that $d \prec d'$, the pair $[d, d']$ is called a *prime interval*. The set of all prime intervals is denoted as $I(D)$. We write $[c, c'] \leq [d, d']$ if and only if $c = c' \sqcap d \vee d' = c' \sqcup d$. The relation $\sim$ is defined to be a transitive symmetric closure of the relation $\leq$. Note that $\sim$-equivalent prime intervals model one and the same action. Let $[d, d']_\sim$ denote the $\sim$-equivalence class of the prime interval $[d, d']$.

Now we are ready to present the definition of marked Scott domains. Informally, a marked Scott domain is meant to be a prime algebraic, finitary, and coherent Scott domain with the prime intervals modeling two – instantaneous and delayed – types of system actions. The former actions do not require time and are marked by zero, and the latter take one unit of time and are marked by one. It is natural to require that the $\sim$-equivalent prime intervals corresponding to one and the same system action are marked identically.

Definition 4. *A* marked domain *is a triple* $(D, \sqsubseteq, m)$*, where* $(D, \sqsubseteq)$ *is a prime algebraic, finitary, and coherent Scott domain and* $m : I(D) \longrightarrow \{0, 1\}$ *is a marking such that* $[c, c'] \sim [d, d'] \Rightarrow m([c, c']) = m([d, d'])$*.*

Introduce auxiliary notions and notations. For $d, d' \in D$ and $i \in \{0, 1\}$, we write $d \prec^i d'$, if $d \prec d' \wedge m([d, d']) = i$, and $d \preccurlyeq^i d'$, if $d \prec^i d' \vee d = d'$; $\sqsubseteq^i = (\prec^i)^*$; $\downarrow^i d = \{d' \mid d' \sqsubseteq^i d\}$, and $\uparrow^i d = \{d' \mid d \sqsubseteq^i d'\}$; $P^i(D) = \{p \in P(D) \mid \exists d \in D \,.\, m([d, p]) = i\}$. For a finite element $d \in D$ and a covering chain σ having the form $\bot = d_0 \prec^{k_1} d_1 \, \cdots \, d_{n-1} \prec^{k_n} d_n = d$ (the chain is finite as $(D, \sqsubseteq)$ is finitary), define the *norm* of d along σ by $\|d\|_\sigma = \sum_{i=1}^n k_i$. Since $(D, \sqsubseteq)$ is a prime algebraic Scott domain and m respects $\sim$, the value of $\|d\|_\sigma$ does not depend on σ. Therefore, we shall use $\|d\|$ to denote the norm of a finite element d. For a non-finite element $d \in D$, its norm is defined as follows: $\|d\| = sup\{\|d'\| \mid d' \in \downarrow d \cap C(D)\}$. A marked domain $(D, \sqsubseteq, m)$ is said to be *linear* if for any $d \in D$ such that $\|d\| < \infty$, $(\uparrow^1 d, \sqsubseteq^1) \cong (\mathbb{N}, \leq)$; *regular* if for any $d, d' \in D$, $d \uparrow d' \Rightarrow \forall d_1 \in \uparrow^1 d,\ \forall d'_1 \in \uparrow^1 d' \,.\, (d_1 \uparrow d'_1)$.

It is not difficult to see that linear regular marked domains, together with the additive stable mappings [10] preserving $\preccurlyeq^0$ and $\prec^1$, form the category **MDom**.

As shown in [9], marked Scott domains are related with timed event structures via a pair of functors $tpes.mdom$: **TPES** $\rightarrow$ **MDom** and $mdom.tpes$: **MDom** $\rightarrow$ **TPES** defined as follows[4].

For a timed event structure $\mathcal{TE} = (E, \leq, \#, \Delta)$, let $tpes.mdom(\mathcal{TE})$ be $(\mathrm{TConf}(\mathcal{TE}), \longrightarrow, m_{\mathcal{TE}})$, where

$$m\big([(C,\tau),(C',\tau')]\big) = \begin{cases} 0, \text{ if } \ C' \setminus C = \{e\} \wedge \tau' = \tau, \\ 1, \text{ if } \ C' = C \wedge \tau' = \tau + 1. \end{cases}$$

For a marked Scott domain $MD = (D, \sqsubseteq, m) \in$ **MDom**, define $mdom.tpes(MD)$ to be $(E, \leq, \#, \Delta)$, where $E = P^0(D)$, $p \leq p' \iff p \sqsubseteq p$, $p \# p' \iff p \not\uparrow p'$, and $\Delta(p) = \|p\|$.

[4] We do not specify how $tpes.mdom$ and $mdom.tpes$ act on morphisms since it is not essential to this paper.

Theorem 3. *[9]. The functors tpes.mdom and mdom.tpes constitute an equivalence between the categories* **TPES** *and* **MDom***.*

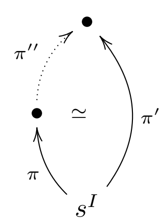
Theorems 1, 2 and 3 yield the following corollary.

Theorem 4. *The functor* $\hookrightarrow \circ tpes.totsi \circ mdom.tpes$: **MDom** $\to$ **TTSI** *is left adjoint to the functor* $tpes.mdom \circ totsi.tpes \circ ttsi.totsi$: **TTSI** $\to$ **MDom**. *Moreover, this adjunction is a coreflection.*

5 Direct Characterization

In this section, we establish some relationships between timed transition systems with independence and marked Scott domains in a direct way.

We start with introducing auxiliary notations. For a transition system with independence $TI = (S, s^I, L, Tran, I)$ and computations $\pi, \pi' \in \mathrm{Comp}^0(TI)$, we write $\pi \trianglelefteq \pi'$ iff there exists a path π'' such that $\pi\pi'' \simeq \pi'$:



For possibly infinite computations $\pi, \pi' \in \mathrm{Comp}(TI)$, let $\pi \trianglelefteq \pi'$ iff for every finite prefix $\bar{\pi}$ of π there exists a finite prefix $\bar{\pi}'$ of π' such that $\bar{\pi} \trianglelefteq \bar{\pi}$. It is straightforward to check that $\trianglelefteq$ is a partial order on $\mathrm{Comp}(TI)$. Specify a partial order on timed computations as follows: $\Pi = (\pi, \tau) \trianglelefteq_\tau \Pi' = (\pi', \tau')$ iff $\pi \trianglelefteq \pi' \wedge \tau \leq \tau'$. Define a partial order $\sqsubseteq$ on the $\simeq_\tau$-equivalence classes of timed computations as follows: $[\Pi]_\tau \sqsubseteq [\Pi']_\tau$ iff $\Pi \trianglelefteq_\tau \Pi'$.

Lemma 4. $(\mathrm{TComp}_{\simeq_\tau}(TTI), \sqsubseteq)$ *is a finitary ω-algebraic dcpo. Moreover,*

$$C((\mathrm{TComp}_{\simeq_\tau}(TTI), \sqsubseteq)) = \mathrm{TComp}^0_{\simeq_\tau}(TTI).$$

In order to directly relate timed transition systems with independence and marked Scott domains, we construct a mapping $ttsi.mdom'$: **TTSI** $\to$ **MDom**. Before doing so, consider a prime interval $\big[[\Pi = (\pi, \tau)]_\tau, [\Pi' = (\pi', \tau')]_\tau\big]$ in $(\mathrm{TComp}_{\simeq_\tau}(TTI), \sqsubseteq)$. It is not difficult to check that either $\pi' \simeq \pi \wedge \tau' = \tau + 1$ or $\pi' \simeq \pi t \wedge \tau' = \tau$ for some transition t. Define a map $m_{TTI} : I((\mathrm{TComp}_{\simeq_\tau}(TTI), \sqsubseteq)) \to \{0, 1\}$ as follows:

$$m_{TTI}(\big[[\Pi]_\tau, [\Pi']_\tau\big]) = \begin{cases} 0, \text{ if } \tau = \tau', \\ 1, \text{ otherwise.} \end{cases}$$

Let $ttsi.mdom'(TTI) = (\mathrm{TComp}_{\simeq_\tau}(TTI), \sqsubseteq, m_{TTI})$, for any timed transition system with independence TTI.

Proposition 2. *$ttsi.mdom'$ can be extended to a functor $ttsi.mdom' : \mathbf{TTSI} \to \mathbf{MDom}$ isomorphic to $ttsi.mdom = tpes.mdom \circ ottsi.tpes \circ ttsi.ottsi$.*

At last, we are ready to state the fact which is the last main result of this paper and that provides a direct characterisation.

Theorem 5. *$ttsi.mdom'$ is right adjoint to $mdom.ttsi = tpes.mdom \circ ottsi.tpes \circ ttsi.ottsi$. Moreover, this adjunction is a coreflection.*

6 Conclusion

We have defined and studied a timed extension of a well-known "true concurrent" model of transition systems with independence and have shown that there exists a chain of coreflections between a category of the model and a category of marked Scott domains as well as a direct translation. The diagram below summarises the established relationships:

$$\begin{tikzcd}
\mathbf{TTSI} \arrow[r, hook, shift left, "ttsi.totsi"] \arrow[r, phantom, "\top"] \arrow[dr, dashed, shift left, "ttsi.mdom"] \arrow[dr, phantom, "\top"] \arrow[dr, bend right, "ttsi.mdom'"'] & \mathbf{ToTSI} \arrow[l, shift left] \arrow[r, shift left, "totsi.tpes"] \arrow[r, phantom, "\top"] & \mathbf{TPES} \arrow[l, shift left, "tpes.totsi"] \arrow[dl, shift left, "tpes.mdom"] \\
& \mathbf{MDom} \arrow[ul, dashed, shift left, "mdom.ttsi"] \arrow[ur, shift left, "mdom.tpes"] \arrow[ur, phantom, "\cong"] &
\end{tikzcd}$$

References

1. Bednarczyk, M.: Categories of asynchronous systems. PhD thesis, University of Sussex, UK (1987)
2. Shields, M.: Concurrent Machines. The Computer Journal **28**(5) (1985) 449–465
3. Sassone, V., Nielsen, M., Winskel, G.: Models for concurrency: towards a classification. Theoretical Computer Science **170**(1-2) (1996) 297–348
4. McLane, S.: Categories for the working mathematician. Graduate Texts in Mathematics. Springer, Berlin (1971)
5. Hildebrandt, T., Sassone, V.: Comparing Transition Systems with Independence and Asynchronous Transition Systems. International Conference on Concurrency Theory (1996) 84–97
6. Hildebrandt, T., Sassone, V.: Transition Systems with Independence and Multi-Arcs. BRICS Report Series RS-97-10, BRICS, Department of Computer Science, University of Aarhus, April (1997)
7. Alur, R., Dill, D.: A theory of timed automat. Theoretical computer science **126**(2) (1994) 183–235
8. Henzinger, T., Manna, Z., Pnueli, A.: Timed transition systems. In: Real-Time: Theory in Practice, Springer (1992) 226–251

9. Virbitskaite, I.B., Dubtsov, R.S.: Semantic domains of timed event structures. Programming and Computer Software **34**(3) (2008) 125–137
10. Winskel, G.: Event structures. Lecture Notes in Computer Science **255** (1987) 325–392

Appendix A: Elements of Category Theory

Here we briefly recall notions from category theory [4] important to this paper. Let $G : \mathbf{B} \to \mathbf{A}$ be a functor between categories $\mathbf{A}$ and $\mathbf{B}$, and let, for each object A of $\mathbf{A}$, there exist an object $F(A)$ of $\mathbf{B}$ and a morphism $\eta_A : A \to G \circ F(A)$ in $\mathbf{A}$ that is universal in the following sense: for any morphism $h : A \to G(B)$ in $\mathbf{A}$, where B is an object of $\mathbf{B}$, there exists a unique morphism $h' : F(A) \to B$ in $\mathbf{B}$ such that $G(h') \circ \eta_A = h$; i.e., the following diagram commutes.

$$
\begin{tikzcd}
A \arrow[d, "\forall h"'] \\
G(B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \arrow[d, "\exists! h'"'] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "h"'] & G \circ F(A) \arrow[dl, "G(h')"] \\
G(B)
\end{tikzcd}
$$

In this case, we say that there exists an adjunction from $\mathbf{A}$ to $\mathbf{B}$, and the family of morphisms $\{\eta_A \mid A \in \mathbf{A}\}$ is said to be a unit of this adjunction. Then, F can be extended to a functor by assuming that, for any morphism $h : A \to A'$ in $\mathbf{A}$, $F(h) : F(A) \to F(A')$ is a unique morphism in $\mathbf{B}$ such that $G \circ F(h) \circ \eta_A = \eta_{A'} \circ h$. In this case, F is said to be left adjoint to G (denoted as $F \vdash G$), and G right adjoint to F (denoted as $G \dashv F$). In addition, if η_A is an isomorphism for each A, then the adjunction is called a coreflection. Categories $\mathbf{A}$ and $\mathbf{B}$ are equivalent if F is adjoint to G and both the unit and counit of the adjunction are isomorphisms.