# $\mathcal{SHOIQ}$ with transitive closure of roles is decidable

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**Abstract.** The Semantic Web makes an extensive use of the OWL DL ontology language, underlied by the $\mathcal{SHOIQ}$ description logic, to formalize its resources. In this paper, we propose a decision procedure for this logic extended with the transitive closure of roles in concept axioms, a feature needed in several application domains. To address the problem of consistency in this logic, we introduce a new structure for characterizing models which may have an infinite non-tree-like part.

## 1 Introduction

The ontology language OWL-DL [1] is widely used to formalize data resources on the Semantic Web. This language is mainly based on the description logic $\mathcal{SHOIN}$ which is known to be decidable [2]. Although $\mathcal{SHOIN}$ provides *transitive roles* to model transitivity of relations, we can find several applications in which the *transitive closure of roles*, that is more expressive than transitive roles, is needed. For instance, if we denote by $\mathsf{R}^-$ and $\mathsf{R}^+$ the inverse and transitive closure of a role $\mathsf{R}$ respectively then it is obvious that the concept $\exists\mathsf{R}^+.\forall\mathsf{R}^-.\bot$ is unsatisfiable w.r.t. an empty TBox. If we now substitute $\mathsf{R}^+$ for a transitive role $\mathsf{R}_t$ such that $\mathsf{R} \sqsubseteq \mathsf{R}_t$ (i.e. we substitute each occurrence of $\mathsf{R}^+$ in axioms for $\mathsf{R}_t$) then the concept $\exists\mathsf{R}_t.\forall\mathsf{R}^-.\bot$ is satisfiable. The point is that an instance of $\mathsf{R}^+$ represents a sequence of instances of $\mathsf{R}$ but an instance of $\mathsf{R}_t$ corresponds to a sequence of instances of *itself*.

In this paper, we consider an extension of $\mathcal{SHOIQ}$ by enabling transitive closure of roles in concept axioms. In the general case, transitive closure is not expressible in the first order logic [3], the logic from which DL is a sublanguage, while the second order logic is sufficiently expressive to do so.

In the DL literature ([4]; [5]), there have been works dealing with transitive closure of roles. Recently, Ortiz [5] has proposed an algorithm for deciding consistency in the logic $\mathcal{ALCQI}b^+_{reg}$ which allows for transitive closure of roles. However, nominals are disallowed in this logic. It is known that reasoning with a DL including number restrictions, inverse roles, nominals and transitive closure of roles is hard. The reason for this is that there exists an ontology in that DL whose models have an *infinite* non-tree-like part. Calvanese *et al.* [6] have presented an automata-based technique for dealing with the logic $\mathcal{ZOIQ}$ that includes transitive closure of roles, and showed that the sublogics $\mathcal{ZIQ}$, $\mathcal{ZOQ}$ and $\mathcal{ZOI}$ are decidable. To obtain this result, the authors have introduced the *quasi-forest model property* to characterize models of ontologies in these sublogics.

Although they are very expressive, none of these sublogics includes $\mathcal{SHOIQ}$ with transitive closure of roles, namely $\mathcal{SHOIQ}_{(+)}$. The following example[3], noted $\mathcal{K}_1$, shows that there is an ontology in $\mathcal{SHOIQ}_{(+)}$ which does not enjoy the quasi-forest model property. We consider the following axioms:

(1) $\{o\} \sqsubseteq \mathsf{A}$; $\mathsf{A} \sqcap \mathsf{B} \sqsubseteq \bot$; $\mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{A} \sqcap \exists \mathsf{R}'.\mathsf{B}$; $\mathsf{B} \sqsubseteq \exists \mathsf{S}^+.\{o\}$

(2) $\{o\} \sqsubseteq \forall \mathsf{X}^-.\bot$; $\top \sqsubseteq\ \leq 1\, \mathsf{X}.\top$; $\top \sqsubseteq\ \leq 1\, \mathsf{X}^-.\top$ where $\mathsf{X} \in \{\mathsf{R}, \mathsf{R}', \mathsf{S}\}$

Figure 1 shows an infinite non-tree-like model of $\mathcal{K}_1$. In fact, each individual $x$ that satisfies $\exists \mathsf{S}^+.\{o\}$ must have two distinct paths from $x$ to the individual satisfying nominal $o$. Intuitively, we can see that (i) such a $x$ must satisfy $\exists \mathsf{S}^+.\{o\}$ and $\mathsf{B}$, (ii) an individual satisfying $\mathsf{B}$ must connect to another individual satisfying $\mathsf{A}$ which must have a $R$-path to nominal $o$, and (iii) two concepts $\mathsf{A}$ and $\mathsf{B}$ are disjoint.

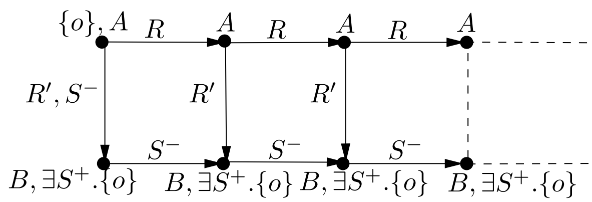


**Fig. 1.** An infinite non tree-like model of $\mathcal{K}_1$

This example shows that methods ([7], [8], [6]) based on the hypothesis which says that if an ontology is consistent it has a *quasi-forest model*, could fail to address the problem of consistency in a DL including simultaneously $\mathcal{O}$ (nominals), $\mathcal{I}$ (inverse roles), $\mathcal{Q}$ (number restrictions) and transitive closure of roles.

In this paper, we propose a decision procedure for the problem of consistency in $\mathcal{SHOIQ}$ with transitive closure of roles in concept axioms. The underlying idea of our algorithm is founded on the *star-type* and *frame* notions introduced by Pratt-Hartmann [9]. This technique uses star-types to represent individuals and "tiles" them together to form a frame for representing a model. For each star-type $\sigma$, we maintain a function $\delta(\sigma)$ which stores the number of individuals satisfying this star-type. To obtain termination condition, we introduce two additional structures into a frame : (i) the first one, namely *cycles*, describes duplicate parts of a model resulting from interactions of logic constructors in $\mathcal{SHOIQ}$, (ii) the second one, namely *blocking-blocked cycles*, describes parts of a model bordered by cycles which allow a frame to satisfy transitive closure of roles occurring in concepts of the form $\exists R^+.C$.

## 2 The Description Logic $\mathcal{SHOIQ}_{(+)}$

In this section, we present the syntax, the semantics and main inference problems of $\mathcal{SHOIQ}_{(+)}$. In addition, we introduce a tableau structure for $\mathcal{SHOIQ}_{(+)}$, which allows us to represent a model of a $\mathcal{SHOIQ}_{(+)}$ knowledge base.

[3] This example is initially proposed by Sebastian Rudolph from an informal discussion

**Definition 1.** *Let $\mathbf{R}$ be a non-empty set of* role names *and $\mathbf{R}_+ \subseteq \mathbf{R}$ be a set of transitive role names. We use $\mathbf{R}_\mathsf{I} = \{P^- \mid P \in \mathbf{R}\}$ to denote a set of inverse roles, and $\mathbf{R}_\oplus = \{Q^+ \mid Q \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$ to denote a set of transitive closure of roles. Each element of $\mathbf{R} \cup \mathbf{R}_\mathsf{I} \cup \mathbf{R}_\oplus$ is called a $\mathcal{SHOIQ}_{(+)}$-role. A* role inclusion axiom *is of the form $R \sqsubseteq S$ for two $\mathcal{SHOIQ}_{(+)}$-roles $R$ and $S$ such that $R \notin \mathbf{R}_\oplus$ and $S \notin \mathbf{R}_\oplus$. A* role hierarchy *$\mathcal{R}$ is a finite set of role inclusion axioms. An interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of a non-empty set $\Delta^\mathcal{I}$ (*domain*) and a function $\cdot^\mathcal{I}$ which maps each role name to a subset of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$ such that*

$$R^{-\mathcal{I}} = \{\langle x, y\rangle \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid \langle y, x\rangle \in R^\mathcal{I}\} \text{ for all } R \in \mathbf{R},$$
$$\langle x, z\rangle \in S^\mathcal{I}, \langle z, y\rangle \in S^\mathcal{I} \text{ implies } \langle x, y\rangle \in S^\mathcal{I} \text{ for each } S \in \mathbf{R}_+, \text{ and}$$
$$(Q^+)^\mathcal{I} = \bigcup_{n>0} (Q^n)^\mathcal{I} \text{ with } (Q^1)^\mathcal{I} = Q^\mathcal{I},$$
$$(Q^n)^\mathcal{I} = \{\langle x, y\rangle \in (\Delta^\mathcal{I})^2 \mid \exists z \in \Delta^\mathcal{I}, \langle x, z\rangle \in (Q^{n-1})^\mathcal{I}, \langle z, y\rangle \in Q^\mathcal{I}\} \text{ for } Q^+ \in \mathbf{R}_\oplus$$

$*$ *An interpretation $\mathcal{I}$ satisfies a role hierarchy $\mathcal{R}$ if $R^\mathcal{I} \subseteq S^\mathcal{I}$ for each $R \sqsubseteq S \in \mathcal{R}$. Such an interpretation is called a* model *of $\mathcal{R}$, denoted by $\mathcal{I} \models \mathcal{R}$. To simplify notations for nested inverse roles and transitive closures of roles, we define two functions $\cdot^\ominus$ and $\cdot^\oplus$ as follows:*

$$R^\ominus = \begin{cases} R^- & \text{if } R \in \mathbf{R};\\ S & \text{if } R = S^- \text{ and } S \in \mathbf{R};\\ (S^-)^+ & \text{if } R = S^+,\ S \in \mathbf{R},\\ S^+ & \text{if } R = (S^-)^+,\ S \in \mathbf{R} \end{cases} \quad R^\oplus = \begin{cases} R^+ & \text{if } R \in \mathbf{R};\\ S^+ & \text{if } R = (S^+)^+ \text{ and } S \in \mathbf{R};\\ (S^-)^+ & \text{if } R = S^- \text{ and } S \in \mathbf{R};\\ (S^-)^+ & \text{if } R = (S^+)^- \text{ and } S \in \mathbf{R} \end{cases}$$

$*$ *A relation $\underline{\circledast}$ is defined as the transitive-reflexive closure $\mathcal{R}^+$ of $\sqsubseteq$ on $\mathcal{R} \cup \{R^\ominus \sqsubseteq S^\ominus \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{R^\oplus \sqsubseteq S^\oplus \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{Q \sqsubseteq Q^\oplus \mid Q \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$. We define a function* $\mathsf{Trans}(R)$ *which returns* true *iff there is some $Q \in \mathbf{R}_+ \cup \{P^\ominus \mid P \in \mathbf{R}_+\} \cup \{P^\oplus \mid P \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$ such that $Q \underline{\circledast} R \in \mathcal{R}^+$. A role $R$ is called* simple *w.r.t. $\mathcal{R}$ if* $\mathsf{Trans}(R) = \mathsf{false}$.

The reason for the introduction of two functions $\cdot^\ominus$ and $\cdot^\oplus$ in Definition 1 is that they avoid using $R^{--}$ and $R^{++}$. Moreover, it remains a unique nested case $(R^-)^+$. According to Definition 1, axiom $R \sqsubseteq Q^\oplus$ is not allowed in a role hierarchy $\mathcal{R}$ since this may lead to undecidability [10]. Notice that the closure $\mathcal{R}^+$ may contain $R \sqsubseteq Q^\oplus$ if $R \sqsubseteq Q$ belongs to $\mathcal{R}^+$.

**Definition 2 (terminology).** *Let $\mathbf{C}$ be a non-empty set of* concept names *with a non-empty subset $\mathbf{C}_o \subseteq \mathbf{C}$ of nominals. The set of $\mathcal{SHOIQ}_{(+)}$-concepts is inductively defined as the smallest set containing all $C$ in $\mathbf{C}$, $\top$, $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$, $(\leq n\, S.C)$ and $(\geq n\, S.C)$ where $n$ is a positive integer, $C$ and $D$ are $\mathcal{SHOIQ}_{(+)}$-concepts, $R$ is an $\mathcal{SHOIQ}_{(+)}$-role and $S$ is a simple role w.r.t. a role hierarchy. We denote $\bot$ for $\neg\top$. The interpretation function $\cdot^\mathcal{I}$ of an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ maps each concept name to a subset of $\Delta^\mathcal{I}$ such that $\top^\mathcal{I} = \Delta^\mathcal{I}$, $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$, $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$, $(\neg C)^\mathcal{I} = \Delta^\mathcal{I} \backslash C^\mathcal{I}$, $|\{o^\mathcal{I}\}| = 1$ for all $o \in \mathbf{C}_o$, $(\exists R.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \exists y \in \Delta^\mathcal{I}, \langle x, y\rangle \in R^\mathcal{I} \wedge y \in C^\mathcal{I}\}$, $(\forall R.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \forall y \in \Delta^\mathcal{I}, \langle x, y\rangle \in R^\mathcal{I} \Rightarrow y \in C^\mathcal{I}\}$, $(\geq n\, S.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid |\{y \in C^\mathcal{I} \mid \langle x, y\rangle \in S^\mathcal{I}| \geq n\}$, $(\leq n\, S.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid |\{y \in C^\mathcal{I} \mid \langle x, y\rangle \in S^\mathcal{I}| \leq n\}$ where $|S|$ is denoted for the cardinality of a set $S$. An axiom $C \sqsubseteq D$ is called a general concept inclusion (GCI)* *where $C, D$ are $\mathcal{SHOIQ}_{(+)}$-concepts (possibly complex), and a finite set of GCIs is called a terminology $\mathcal{T}$. An interpretation $\mathcal{I}$ satisfies a GCI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and $\mathcal{I}$ satisfies a terminology $\mathcal{T}$ if $\mathcal{I}$ satisfies each GCI in $\mathcal{T}$. Such an interpretation is called a* model *of $\mathcal{T}$, denoted by $\mathcal{I} \models \mathcal{T}$. A pair $(\mathcal{T}, \mathcal{R})$ is called a $\mathcal{SHOIQ}_{(+)}$ knowledge base where $\mathcal{R}$ is a $\mathcal{SHOIQ}_{(+)}$ role hierarchy and $\mathcal{T}$ is a $\mathcal{SHOIQ}_{(+)}$ terminology. A knowledge base $(\mathcal{T}, \mathcal{R})$ is said to be consistent if there is a model $\mathcal{I}$ of both $\mathcal{T}$ and $\mathcal{R}$, i.e., $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{R}$. A concept $C$ is called* satisfiable *w.r.t. $(\mathcal{T}, \mathcal{R})$ iff there is some interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{R}$, $\mathcal{I} \models \mathcal{T}$ and $C^{\mathcal{I}} \neq \emptyset$. Such an interpretation is called a* model *of $C$ w.r.t. $(\mathcal{T}, \mathcal{R})$. A concept $D$* subsumes *a concept $C$ w.r.t. $(\mathcal{T}, \mathcal{R})$, denoted by $C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds in each model $\mathcal{I}$ of $(\mathcal{T}, \mathcal{R})$.* $\lhd$

Since unsatisfiability, subsumption and consistency w.r.t. a $\mathcal{SHOIQ}_{(+)}$ knowledge base can be reduced to each other, it suffices to study knowledge base consistency. For the ease of construction, we assume all concepts to be in *negation normal form* (NNF), i.e., negation occurs only in front of concept names. Any $\mathcal{SHOIQ}_{(+)}$-concept can be transformed to an equivalent one in NNF by using DeMorgan's laws and some equivalences as presented in [11]. According to [12], $\mathsf{nnf}(C)$ can be computed in polynomial time in the size of $C$. For a concept $C$, we denote the $\mathsf{nnf}$ of $C$ by $\mathsf{nnf}(C)$ and the $\mathsf{nnf}$ of $\neg C$ by $\dot{\neg} C$. Let $D$ be a $\mathcal{SHOIQ}_{(+)}$-concept in NNF. We define $\mathsf{cl}(D)$ to be the smallest set that contains all sub-concepts of $D$ including $D$. For a knowledge base $(\mathcal{T}, \mathcal{R})$, we reuse $\mathsf{cl}(\mathcal{T}, \mathcal{R})$, which was introduced by Horrocks *et al.* [7], to denote all sub-concepts occurring in the axioms of $(\mathcal{T}, \mathcal{R})$. We have $\mathsf{cl}(\mathcal{T}, \mathcal{R})$ is bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$ [7]. To translate *star-type* and *frame* structures presented by Pratt-Hartmann (2005) for $\mathcal{C}^2$ into those for $\mathcal{SHOIQ}$, we need to add new sets of concepts, denoted $\mathsf{cl}_1(\mathcal{T}, \mathcal{R})$ and $\mathsf{cl}_2(\mathcal{T}, \mathcal{R})$, to the signature of a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$.

$$\begin{aligned}\mathsf{cl}_1(\mathcal{T}, \mathcal{R}) = &\{\leq mS.C \mid \{(\leq nS.C), (\geq nS.C)\} \cap \mathsf{cl}(\mathcal{T}, \mathcal{R}) \neq \emptyset, 1 \leq m \leq n\} \cup \\ &\{\geq mS.C \mid \{(\leq nS.C), (\geq nS.C)\} \cap \mathsf{cl}(\mathcal{T}, \mathcal{R}) \neq \emptyset, 1 \leq m \leq n\}\end{aligned}$$

For a generating concept $(\geq nS.C)$ and a set $I \subseteq \{0, \cdots, log\ n+1\}$, we denote $\mathscr{C}^I_{(\geq nS.C)} = \bigsqcap_{i \in I} C^i_{(\geq nS.C)} \sqcap \bigsqcap_{j \notin I} \neg C^j_{(\geq nS.C)}$ where $C^i_{(\geq nS.C)}$ are new concept names for $0 \leq i \leq log\ n+1$. We define $\mathsf{cl}_2(\mathcal{T}, \mathcal{R})$ as follows:

$$\begin{aligned}\mathsf{cl}_2(\mathcal{T}, \mathcal{R}) = &\{C^i_{(\geq S.C)} \mid (\geq nS.C) \in \mathsf{cl}(\mathcal{T}, \mathcal{R}) \cup \mathsf{cl}_1(\mathcal{T}, \mathcal{R}), 0 \leq i \leq log\ n+1\} \cup \\ &\{\mathscr{C}^I_{(\geq nS.C)} \mid (\geq nS.C) \in \mathsf{cl}(\mathcal{T}, \mathcal{R}) \cup \mathsf{cl}_1(\mathcal{T}, \mathcal{R}), I \subseteq \{0, \cdots, log\ n+1\}\}\end{aligned}$$

*Remark 1.* If numbers are encoded in binary then the number of new concept names $C^i_{(\geq nS.D)}$ for $0 \leq i \leq log\ n+1$, is bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$ since $n$ is bounded by $\mathcal{O}(2^{|(\mathcal{T}, \mathcal{R})|})$. This implies that $|\mathsf{cl}_2(\mathcal{T}, \mathcal{R})|$ is bounded by $\mathcal{O}(2^{|(\mathcal{T}, \mathcal{R})|})$. Note that two concepts $\mathscr{C}^I_{(\geq nS.C)}$ and $\mathscr{C}^J_{(\geq nS.C)}$ are disjoint for all $I, J \subseteq \{0, \cdots, log\ n+1\}$, $I \neq J$. The concepts $\mathscr{C}_{(\exists S.C)}$ and $\mathscr{C}^I_{(\geq nS.C)}$ will be used for building chromatic star-types. This notion will be clarified after introducing the frame structure (Definition 5).

Finally, we denote $\mathbf{CL}(\mathcal{T}, \mathcal{R}) = \mathsf{cl}(\mathcal{T}, \mathcal{R}) \cup \mathsf{cl}_1(\mathcal{T}, \mathcal{R}) \cup \mathsf{cl}_2(\mathcal{T}, \mathcal{R})$, and use $\mathbf{R}(\mathcal{T}, \mathcal{R})$ to denote the set of all role names occurring in $\mathcal{T}, \mathcal{R}$ with their inverse. The definition of $\mathbf{CL}(\mathcal{T}, \mathcal{R})$ is inspired from the Fischer-Ladner closure that was introduced in [13]. The closure $\mathbf{CL}(\mathcal{T}, \mathcal{R})$ contains not only sub-concepts syntactically obtained from $\mathcal{T}$

but also sub-concepts that are semantically derived from $\mathcal{T}$ w.r.t. $\mathcal{R}$. For instance, if $\forall S.C$ is a sub-concept from $\mathcal{T}$ and $R \boxtimes S \in \mathcal{R}$ then $\forall R.C \in \mathbf{CL}(\mathcal{T}, \mathcal{R})$.

To describe a model of a $\mathcal{SHOIQ}_{(+)}$ knowledge base in a more intuitive way, we use a tableau structure that expresses semantic constraints resulting directly from the logic constructors in $\mathcal{SHOIQ}_{(+)}$. A tableau definition for $\mathcal{SHOIQ}_{(+)}$ can be found in [14].

## 3 A Decision Procedure For $\mathcal{SHOIQ}_{(+)}$

This section starts by translating *star-type* and *frame* structures presented by Pratt-Hartmann (2005) for $\mathcal{C}^2$ into those for $\mathcal{SHOIQ}_{(+)}$.

**Definition 3 (star-type).** *Let* $(\mathcal{T}, \mathcal{R})$ *be a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base. A* star-type *is a pair* $\sigma = \langle \lambda(\sigma), \xi(\sigma) \rangle$, *where* $\lambda(\sigma) \in 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})}$ *is called* core label, $\xi(\sigma) = (\langle r_1, l_1 \rangle, \cdots, \langle r_d, l_d \rangle)$ *is a* $d$*-tuple over* $2^{\mathbf{R}(\mathcal{T},\mathcal{R})} \times 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})}$. *A pair* $\langle r, l \rangle$ *is a* ray *of* $\sigma$ *if* $\langle r, l \rangle = \langle r_i, l_i \rangle$ *for some* $1 \leq i \leq d$. *We use* $\langle r(\rho), l(\rho) \rangle$ *to denote a ray* $\rho = \langle r, l \rangle$ *where* $r(\rho) = r$ *and* $l(\rho) = l$.

- *A star-type* $\sigma$ *is* nominal *if* $o \in \lambda(\sigma)$ *for some* $o \in \mathbf{C}_o$.
- *A star-type* $\sigma$ *is* chromatic *if* $\rho \neq \rho'$ *implies* $l(\rho) \neq l(\rho')$ *for two rays* $\rho, \rho'$ *of* $\sigma$. *When a star-type* $\sigma$ *is chromatic,* $\xi(\sigma)$ *can be considered as a set of rays.*
- *Two star-types* $\sigma, \sigma'$ *are* equivalent *if* $\lambda(\sigma) = \lambda(\sigma')$, *and there is a bijection* $\pi$ *between* $\xi(\sigma)$ *and* $\xi(\sigma')$ *such that* $\pi(\rho) = \rho'$ *implies* $r(\rho') = r(\rho)$ *and* $l(\rho') = l(\rho)$.

*We denote* $\Sigma$ *for the set of all star-types for* $(\mathcal{T}, \mathcal{R})$. $\lhd$

Note that for a chromatic star-type $\sigma$, $\xi(\sigma)$ can be considered as a set of rays since rays are distinct and not ordered. We can think of a star-type $\sigma$ as the set of individuals $x$ satisfying all concepts in $\lambda(\sigma)$, and each ray $\rho$ of $\sigma$ corresponds to a "neighbor" individual $x_i$ of $x$ such that $r(\rho)$ is the label of the link between $x$ and $x_i$; and $x_i$ satisfies all concepts in $l(\rho)$. In this case, we say that $x$ *satisfies* $\sigma$.

**Definition 4 (valid star-type).** *Let* $(\mathcal{T}, \mathcal{R})$ *be a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base. Let* $\sigma$ *be a star-type for* $(\mathcal{T}, \mathcal{R})$ *where* $\sigma = \langle \lambda(\sigma), \xi(\sigma) \rangle$. *The star-type* $\sigma$ *is* valid *if* $\sigma$ *is chromatic and the following conditions are satisfied:*

1. *If* $C \sqsubseteq D \in \mathcal{T}$ *then* $\mathsf{nnf}(\neg C \sqcup D) \in \lambda(\sigma)$;
2. $\{A, \neg A\} \not\subseteq \lambda$ *for every concept name* $A$ *where* $\lambda = \lambda(\sigma)$ *or* $\lambda = l(\rho)$ *for each* $\rho \in \xi(\sigma)$;
3. *If* $C_1 \sqcap C_2 \in \lambda(\sigma)$ *then* $\{C_1, C_2\} \subseteq \lambda(\sigma)$;
4. *If* $C_1 \sqcup C_2 \in \lambda(\sigma)$ *then* $\{C_1, C_2\} \cap \lambda(\sigma) \neq \emptyset$;
5. *If* $\exists R.C \in \lambda(\sigma)$ *then there is some ray* $\rho \in \xi(\sigma)$ *such that* $C \in l(\rho)$ *and* $R \in r(\rho)$;
6. *If* $(\leq nS.C) \in \lambda(\sigma)$ *and there is some ray* $\rho \in \xi(\sigma)$ *such that* $S \in r(\rho)$ *then* $C \in l(\rho)$ *or* $\dot\neg C \in l(\rho)$;
7. *If* $(\leq nS.C) \in \lambda(\sigma)$ *and there is some ray* $\rho \in \xi(\sigma)$ *such that* $C \in l(\rho)$ *and* $S \in r(\rho)$ *then there is some* $1 \leq m \leq n$ *such that* $\{(\leq mS.C), (\geq mS.C)\} \subseteq \lambda(\sigma)$;
8. *For each ray* $\rho \in \xi(\sigma)$, *if* $R \in r(\rho)$ *and* $R \boxtimes S$ *then* $S \in r(\rho)$;

9. *If* $\forall R.C \in \lambda(\sigma)$ *and* $R \in r(\rho)$ *for some ray* $\rho \in \xi(\sigma)$ *then* $C \in l(\rho)$*;*
10. *If* $\forall R.D \in \lambda(\sigma)$*,* $S \sqsubseteq^* R$*,* $\mathsf{Trans}(S)$ *and* $R \in r(\rho)$ *for some ray* $\rho \in \xi(\sigma)$ *then* $\forall S.D \in l(\rho)$*;*
11. *If* $\forall Q^{\oplus}.C \in \lambda(\sigma)$*,* $R \sqsubseteq^* Q$ *and* $R \in r(\rho)$ *for some ray* $\rho \in \xi(\sigma)$ *then* $\forall Q^{\oplus}.C \in l(\rho)$*;*
12. *If* $\exists Q^{\oplus}.C \in \lambda(\sigma)$ *then* $(\exists Q.C \sqcup \exists Q.\exists Q^{\oplus}.C) \in \lambda(\sigma)$*;*
13. *If* $(\geq nS.C) \in \lambda(\sigma)$ *then there are* $n$ *distinct rays* $\rho_1, \cdots, \rho_n \in \xi(\sigma)$ *such that* $\{C, \mathscr{C}^{I_i}_{(\geq nS.C)}\} \subseteq l(\rho_i)$*,* $S \in r(\rho_i)$ *for all* $1 \leq i \leq n$*; and* $I_j, I_k \subseteq \{0, \cdots, \log n + 1\}$*,* $I_j \neq I_k$ *for all* $1 \leq j < k \leq n$*.*
14. *If* $(\leq nS.C) \in \lambda(\sigma)$ *and there do not exist* $n+1$ *rays* $\rho_0, \cdots, \rho_n \in \xi(\sigma)$ *such that* $C \in l(\rho_i)$ *and* $S \in r(\rho_i)$ *for all* $0 \leq i \leq n$*.* $\lhd$

Roughly speaking, a star-type $\sigma$ is valid if each individual $x$ satisfies *semantically* all concepts in $\lambda(\sigma)$. In fact, each condition in Definition 4 represents the semantics of a constructor in $\mathcal{SHOIQ}_{(+)}$ except for transitive closure of roles. From valid star-types, we can "tile" a model instead of using expansion rules for generating nodes as described in tableau algorithms. Before presenting how to "tile" a model from star-types, we need some notation that will be used in the remainder of the paper.

**Notation 1** *We call* $\mathcal{P} = \langle(\sigma_1, \rho_1, d_1), \cdots, (\sigma_k, \rho_k, d_k)\rangle$ *a* sequence *where* $\sigma_i \in \Sigma$*,* $\rho_i \in \xi(\sigma_i)$ *and* $d_i \in \mathbb{N}$ *for* $1 \leq i \leq k$*.*

- $\mathsf{tail}(\mathcal{P}) = (\sigma_k, \rho_k, d_k)$*,* $\mathsf{tail}_\sigma(\mathcal{P}) = \sigma_k$*,* $\mathsf{tail}_\rho(\mathcal{P}) = \rho_k$*,* $\mathsf{tail}_\delta(\mathcal{P}) = d_k$ *and* $|\mathcal{P}| = k$*. We denote* $\mathcal{L}(\mathcal{P}) = \lambda(\mathsf{tail}_\sigma(\mathcal{P}))$*.*
- $\mathsf{p}^i(\mathcal{P}) = (\sigma_i, \rho_i, d_i)$*,* $\mathsf{p}^i_\sigma(\mathcal{P}) = \sigma_i$*,* $\mathsf{p}^i_\rho(\mathcal{P}) = \rho_i$ *and* $\mathsf{p}^i_\delta(\mathcal{P}) = d_i$ *for each* $1 \leq i \leq k$*.*
- *an operation* $\mathsf{add}(\mathcal{P}, (\sigma, \rho, d))$ *extends* $\mathcal{P}$ *to a new sequence with* $\mathsf{add}(\mathcal{P}, (\sigma, \rho, d)) = \langle \mathcal{P}, (\sigma, \rho, d)\rangle$*.*

**Definition 5 (frame).** *Let* $(\mathcal{T}, \mathcal{R})$ *be a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base. A* frame *for* $(\mathcal{T}, \mathcal{R})$ *is a tuple* $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta\rangle$*, where*

1. $\mathcal{N}$ *is a set of valid star-types such that* $\sigma$ *is not equivalent to* $\sigma'$ *for all* $\sigma, \sigma' \in \mathcal{N}$*;*
2. $\mathcal{N}_o \subseteq \mathcal{N}$ *is a set of nominal star-types;*
3. $\Omega$ *is a function that maps each pair* $(\sigma, \rho)$ *with* $\sigma \in \mathcal{N}$ *and* $\rho \in \xi(\sigma)$ *to a sequence* $\Omega(\sigma, \rho) = \langle(\sigma_1, \rho_1, d_1), \cdots, (\sigma_m, \rho_m, d_m)\rangle$ *with* $\sigma_i \in \mathcal{N}$*,* $\rho_i \in \xi(\sigma_i)$*,* $d_i \in \mathbb{N}$ *for* $1 \leq i \leq m$ *such that for each* $\sigma_i$ *with* $1 \leq i \leq m$*, it holds that* $l(\rho) = \lambda(\sigma_i)$*,* $l(\rho_i) = \lambda(\sigma)$ *and* $r(\rho_i) = r^-(\rho)$ *where* $r^-(\rho) = \{R^\ominus \mid R \in r(\rho)\}$*.*
4. $\delta$ *is a function* $\delta : \mathcal{N} \rightarrow \mathbb{N}$*. By abuse of notation, we also use* $\delta$ *to denote a function which maps each pair* $(\sigma, \rho)$ *with* $\sigma \in \mathcal{N}$ *and* $\rho \in \xi(\sigma)$ *into a number in* $\mathbb{N}$*, i.e.,* $\delta(\sigma, \rho) \in \mathbb{N}$*.* $\lhd$

The frame structure, as introduced in Definition 5, allows us to compress individuals of a model into star-types. For each star-type $\sigma$ and each ray $\rho \in \xi(\sigma)$, a list $\Omega(\sigma, \rho)$ of triples $(\sigma_i, \rho_i, d_i)$ with $\rho_i \in \xi(\sigma_i)$ is maintained where $\sigma_i$ is a "neighbor" star-type of $\sigma$ via $\rho \in \xi(\sigma)$, and $d_i$ indicates the $d_i$-th "layer" of rays of $\sigma_i$. We can think a layer of rays of $\sigma_i$ as an individual that connects to its neighbor individuals via the rays of $\sigma_i$. The following definition presents how to connect such layers to form paths in a frame.

**Definition 6 (path).** *Let* $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta\rangle$ *be a frame for a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base* $(\mathcal{T}, \mathcal{R})$*. A* path *is inductively defined as follows:*

1. *A sequence $\langle \emptyset, (\sigma, \rho, 1)\rangle$ is a path if $\sigma \in \mathcal{N}_o$ and $\rho \in \xi(\sigma)$;*
2. *A sequence $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ with $\mathcal{P} \neq \emptyset$ and $\mathsf{tail}(\mathcal{P}) = (\sigma_0, \rho_0, d_0)$, is a path if $(\sigma, \rho, d) = \mathsf{p}^{d_0}(\Omega(\sigma_0, \rho'))$ for each $\rho' \neq \rho_0$. In this case, we say that $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ is the $\rho'$-*neighbor *of $\mathcal{P}$, and two paths $\mathcal{P}$, $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ are neighbors.*
   *Additionally, if $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ is a $\rho'$-neighbor of $\mathcal{P}$ and $Q \in r(\rho')$ then $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ is a $Q$-neighbor of $\mathcal{P}$. In this case, we say that $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ is a $Q$-neighbor of $\mathcal{P}$, or $\mathcal{P}$ is a $Q^{\ominus}$-neighbor of $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$.*

*We define $\mathcal{P} \sim \mathcal{P}'$ if $\mathsf{tail}_\sigma(\mathcal{P}) = \mathsf{tail}_\sigma(\mathcal{P}')$ and $\mathsf{tail}_\delta(\mathcal{P}) = \mathsf{tail}_\delta(\mathcal{P}')$. Since $\sim$ is an equivalence relation over the set of all paths, we use $\mathscr{P}$ to denote the set of all equivalence classes $[\mathcal{P}]$ of paths in $\mathcal{F}$. For $[\mathcal{P}], [\mathcal{Q}] \in \mathscr{P}$, we define:*

1. *$[\mathcal{P}]$ is a neighbor ($\rho'$-neighbor) of $[\mathcal{Q}]$ if there are $\mathcal{P}' \in [\mathcal{P}]$ and $\mathcal{Q}' \in [\mathcal{Q}]$ such that $\mathcal{Q}'$ is a neighbor ($\rho'$-neighbor) of $\mathcal{P}'$;*
2. *$[\mathcal{Q}]$ is a reachable path of $[\mathcal{P}]$ if there are $[\mathcal{P}_1], \cdots, [\mathcal{P}_n] \in \mathscr{P}$ such that $[\mathcal{P}_{i+1}]$ is a neighbor of $[\mathcal{P}_i]$ for all $1 \leq i < n$ where $[\mathcal{P}_1] = [\mathcal{P}]$ and $[\mathcal{Q}] = [\mathcal{P}_n]$.*
3. *$[\mathcal{Q}]$ is a $Q$-neighbor of $[\mathcal{P}]$ if there are $\mathcal{P}' \in [\mathcal{P}]$ and $\mathcal{Q}' \in [\mathcal{Q}]$ such that $\mathcal{Q}'$ is a $Q$-neighbor of $\mathcal{P}'$, or $\mathcal{P}'$ is a $Q^{\ominus}$-neighbor of $\mathcal{Q}'$;*
4. *$[\mathcal{Q}]$ is a $Q$-reachable path of $[\mathcal{P}]$ if there are $[\mathcal{P}_1], \cdots, [\mathcal{P}_n] \in \mathscr{P}$ such that $[\mathcal{P}_{i+1}]$ is a $Q$-neighbor of $[\mathcal{P}_i]$ for all $1 \leq i < n$ where $[\mathcal{P}_1] = [\mathcal{P}]$ and $[\mathcal{Q}] = [\mathcal{P}_n]$.* $\lhd$

Note that for two paths $\mathcal{P}, \mathcal{P}'$ with $\mathsf{tail}_\rho(\mathcal{P}) \neq \mathsf{tail}_\rho(\mathcal{P}')$, we have $\mathcal{P} \sim \mathcal{P}'$ if $\mathsf{tail}_\sigma(\mathcal{P}) = \mathsf{tail}_\sigma(\mathcal{P}')$ and $\mathsf{tail}_\delta(\mathcal{P}) = \mathsf{tail}_\delta(\mathcal{P}')$. This does not allow for extending $\mathsf{tail}_\rho(\mathcal{P})$ to $\mathsf{tail}_\rho([\mathcal{P}])$. As a consequence, there may be several "predecessors" of an equivalence class $[\mathcal{P}]$. However, we can define $\mathsf{tail}_\sigma([\mathcal{P}]) = \mathsf{tail}_\sigma(\mathcal{P})$, $\mathsf{tail}_\delta([\mathcal{P}]) = \mathsf{tail}_\delta(\mathcal{P})$ and $\mathcal{L}([\mathcal{P}]) = \mathcal{L}(\mathcal{P})$. In the sequel, we use $\mathcal{P}$ instead of $[\mathcal{P}]$ whenever it is clear from the context.

**Definition 7 (cycle).** *Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta\rangle$ be a frame for a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$ with a set $\mathscr{P}$ of paths in $\mathcal{F}$.*

1. *A* cycle *is a set $\Theta$ of triples $(\mathcal{P}, \rho, \mathcal{Q})$ with $\mathcal{P}, \mathcal{Q} \in \mathscr{P}$ and $\rho \in \xi(\mathsf{tail}_\sigma(\mathcal{P}))$ such that for each $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ the following conditions are satisfied:*
   *(a) $\mathsf{tail}_\delta(\mathcal{P}) > 1$ and $\mathsf{tail}_\delta(\mathcal{Q}) > 1$;*
   *(b) If $\mathcal{P}'$ is the $\rho$-neighbor of $\mathcal{P}$ then $\mathsf{tail}_\sigma(\mathcal{P}') = \mathsf{tail}_\sigma(\mathcal{Q})$;*
   *(c) for each sequence $\mathcal{P}_1, \cdots, \mathcal{P}_n \in \mathscr{P}$ such that $\mathcal{P}_1 = \mathcal{P}$, $\mathcal{P}_2$ is not the $\rho$-neighbor of $\mathcal{P}$, and $\mathcal{P}_{i+1}$ is a neighbor of $\mathcal{P}_i$ for $1 \leq i < n$, there is some $(\mathcal{P}'', \rho'', \mathcal{Q}'') \in \Theta$ such that*
      *i. either $\mathcal{Q}'' = \mathcal{P}_j$, $\mathsf{tail}_\sigma(\mathcal{P}_{j+1}) = \mathsf{tail}_\sigma(\mathcal{P}'')$, $\mathsf{tail}_\delta(\mathcal{P}_{j+1}) \geq \mathsf{tail}_\delta(\mathcal{P}'')$ and $\mathcal{P}_j$ is the $\rho$-neighbor of $\mathcal{P}_{j+1}$ for some $1 < j < n$,*
      *ii. or there are $\mathcal{P}_{n+1}, \cdots, \mathcal{P}_{n+m} \in \mathscr{P}$ with $\mathcal{Q}'' = \mathcal{P}_{n+m-1}$, $\mathsf{tail}_\sigma(\mathcal{P}_{n+m}) = \mathsf{tail}_\sigma(\mathcal{P}'')$, $\mathsf{tail}_\delta(\mathcal{P}_{n+m}) \geq \mathsf{tail}_\delta(\mathcal{P}'')$ and $\mathcal{P}_{i+1}$ is a neighbor of $\mathcal{P}_i$ for $n \leq i < n+m$.*

   *In this case, we say that $\mathcal{Q}$ is* cycled *by $\mathcal{P}$ via $\rho$.*
2. *A cycle $\Theta'$ is a* reachable *cycle of $\Theta$ if for each $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ and for each sequence $\mathcal{P}_1, \cdots, \mathcal{P}_n \in \mathscr{P}$ such that $\mathcal{P}_1 = \mathcal{P}$, $\mathcal{P}_2$ is not a $\rho$-neighbor of $\mathcal{P}$, and $\mathcal{P}_{i+1}$ is a neighbor of $\mathcal{P}_i$ for $1 \leq i < n$, there is some $(\mathcal{P}'', \rho'', \mathcal{Q}'') \in \Theta'$ such that*

(a) *either* $\mathcal{Q}'' = \mathcal{P}_j$, $\mathsf{tail}_\sigma(\mathcal{P}_{j+1}) = \mathsf{tail}_\sigma(\mathcal{P}'')$, $\mathsf{tail}_\delta(\mathcal{P}_{j+1}) \geq \mathsf{tail}_\delta(\mathcal{P}'')$ *and* $\mathcal{P}_j$ *is a* $\rho$*-neighbor of* $\mathcal{P}_{j+1}$ *for some* $1 < j < n$,

(b) *or there are* $\mathcal{P}_{n+1}, \cdots, \mathcal{P}_{n+m} \in \mathscr{P}$ *with* $\mathcal{Q}'' = \mathcal{P}_{n+m-1}$, $\mathsf{tail}_\sigma(\mathcal{P}_{n+m}) = \mathsf{tail}_\sigma(\mathcal{P}'')$, $\mathsf{tail}_\delta(\mathcal{P}_{n+m}) \geq \mathsf{tail}_\delta(\mathcal{P}'')$ *and* $\mathcal{P}_{i+1}$ *is a neighbor of* $\mathcal{P}_i$ *for* $n \leq i < n + m$. $\lhd$

Note that cycles may encapsulate loops if $\mathsf{tail}_\delta(\mathcal{P}_{j+1}) = \mathsf{tail}_\delta(\mathcal{P}'')$ holds in Conditions 1(c)i and 1(c)ii, Definition 7. Let $\Theta$ be a cycle in a frame. Definition 7 ensures that each reachable path of some path $\mathcal{P}$ with $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ goes through a star-type $\sigma = \mathsf{tail}_\sigma(\mathcal{Q}')$ with some $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta$. Such a cycle, which is similar to blocking-blocked nodes in completion graphs for $\mathcal{SHOIQ}$ [7], allows for "unravelling" infinitely the frame to obtain a model of a KB in $\mathcal{SHOIQ}$ (without transitive closure of roles). This means that we can extend the set $\mathscr{P}$ of paths by adding infinitely paths which lengthen $\mathcal{Q}$ such that $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ and $\mathcal{P}$ is not a neighbor of $\mathcal{Q}$. However, such a cycle structure is not sufficient to represent models of a KB with transitive closure of roles since a concept such as $\exists Q^{\oplus}.D \in \mathcal{L}(\mathcal{P})$ can be satisfied by a $Q$-reachable path $\mathcal{P}'$ of $\mathcal{P}$ which is arbitrarily far from $\mathcal{P}$. There are the following possibilities for an algorithm which builds a frame: (i) the algorithm stops building the frame as soon as a cycle $\Theta$ is detected such that each concept of the form $\exists Q^{\oplus}.D$ occurring in $\mathcal{L}(\mathcal{P})$ for each cycling path $\mathcal{P}$ of $\Theta$ is satisfied, i.e., $\mathcal{P}$ has a $Q$-reachable path $\mathcal{P}'$ with $\in \exists Q.D \in \mathcal{L}(\mathcal{P})$, (ii) despite of several detected cycles, the algorithm continues building the frame until each concept of the form $\exists Q^{\oplus}.D$ occurring in $\mathcal{L}(\mathcal{P})$ is satisfied for each cycling path $\mathcal{P}$ of $\Theta$. If we adopt the first possibility, the completeness of such an algorithm cannot be established since there are models in which paths satisfying concepts of the form $\exists Q^{\oplus}.D$ can spread over several "iterative structures" such as cycles. For this reason, we adopt the second possibility by introducing into frames an additional structure, namely *blocking-blocked cycles*, which determines a sequence of cycles $\Theta_1, \cdots, \Theta_k$ such that $\Theta_{i+1}$ is a reachable cycle of $\Theta_i$. Reachability of cycles allows for "unravelling" the frame between cycled paths $\mathcal{Q}'$ with $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta_k$ and cycling paths $\mathcal{P}$ with $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta_1$.

**Definition 8 (blocking).** *Let* $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ *be a frame for a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base* $(\mathcal{T}, \mathcal{R})$ *with a set* $\mathscr{P}$ *of all paths in* $\mathcal{F}$.

1. *A cycle* $\Theta'$ *is* blockable *by a cycle* $\Theta$ *if* $\Theta'$ *is a reachable cycle of* $\Theta$, *and for each* $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta'$ *there is some* $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ *such that* $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{P}')$, $\mathcal{L}(\mathcal{Q}) = \mathcal{L}(\mathcal{Q}')$ *and* $r(\rho) = r(\rho')$. *In this case, we say that* $\mathcal{Q}'$ *is blockable by* $\mathcal{P}$ *via* $\rho$.
2. *A cycle* $\Theta'$ *is* blocked *by a cycle* $\Theta$ *if there are* $\Theta_1, \cdots, \Theta_k$ *with* $\Theta = \Theta_1$, $\Theta' = \Theta_k$ *such that* $\Theta_{i+1}$ *is blockable by* $\Theta_i$ *for* $1 \leq i < k$, *and for each* $(\lambda, s, \lambda') \in 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})} \times 2^{\mathbf{R}(\mathcal{T},\mathcal{R})} \times 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})}$ *with* $\exists Q^{\oplus}.D \in \lambda$,
   - *if there is some path* $\mathcal{P}_k$ *with* $(\mathcal{P}_k, \rho_k, \mathcal{Q}_k) \in \Theta_k$, $\mathcal{L}(\mathcal{P}_k) = \lambda$, $\mathcal{L}(\mathcal{Q}_k) = \lambda'$, $s = r(\rho_k)$ *such that* $\mathcal{P}_k$ *has no* $Q$*-reachable path* $\mathcal{P}'_k$ *with* $\exists Q.D \in \mathcal{L}(\mathcal{P}'_k)$,
   - *then there is some* $\mathcal{P}_1$ *with* $(\mathcal{P}_1, \rho_1, \mathcal{Q}_1) \in \Theta_1$, $\mathcal{L}(\mathcal{P}_1) = \lambda$, $\mathcal{L}(\mathcal{Q}_1) = \lambda'$, $r(\rho_1) = s$ *such that for each* $\exists P^{\oplus}.C \in \mathcal{L}(\mathcal{P}_1)$,
     - *there is a* $P$*-reachable path* $\mathcal{Q}''$ *of* $\mathcal{P}_1$ *with* $\exists P.C \in \mathcal{L}(\mathcal{Q}'')$, *and*

- *there are two triples $(\mathcal{P}_j, \rho_j, \mathcal{Q}_j) \in \Theta_j$ and $(\mathcal{P}_{j+1}, \rho_{j+1}, \mathcal{Q}_{j+1}) \in \Theta_{j+1}$ for some $1 \leq j < k$,*

*which satisfy that $\mathcal{Q}''$ is a reachable path of $\mathcal{Q}_j$ and $\mathcal{Q}_{j+1}$ is a reachable path of $\mathcal{Q}''$.*

*In this case, we say that $\mathcal{P}_1$ blocks $\mathcal{P}_k$ via $\rho_k$.* $\triangleleft$

According to Definition 8, there are a sequentially reachable cycles between a blocking cycle $\Theta_1$ and a blocked cycle $\Theta_k$, which allows for unravelling the frame between $\Theta_k$ and $\Theta_1$. Condition 2 Definition 8 says that if $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta_k$ and $\mathcal{L}(\mathcal{P})$ contains concepts of the form $\exists Q^{\oplus}.D$ which are not satisfied by reachable paths of $\mathcal{P}$ then there exists some $\mathcal{P}'$ with $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta_1$ which allows for satisfying these concepts $\exists Q^{\oplus}.D$ in $\mathcal{L}(\mathcal{P})$ by unravelling. We would like to note that a path $\mathcal{P}$ is blocked if there is some blocked cycle $\Theta_k$ such that $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta_k$.

**Definition 9 (valid frame).** *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}$ knowledge base. A frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ is* valid *if the following conditions are satisfied:*

1. *For each $o \in \mathbf{C}_o$ there is a unique $\sigma_o \in \mathcal{N}_o$ such that $o \in \lambda(\sigma_o)$ and $\delta(\sigma_o) = 1$;*
2. *For each $\sigma \in \mathcal{N}$, $\sigma$ is valid;*
3. *If $\exists Q^{\oplus}.C \in \lambda(\mathsf{tail}_\sigma(\mathcal{P}_0))$ for some $\mathcal{P}_0 \in \mathscr{P}$ then there are $\mathcal{P}, \mathcal{P}' \in \mathscr{P}$ such that one of the following conditions is satisfied:*
   *(a) $\mathcal{P}_0 = \mathcal{P} = \mathcal{P}'$ and $\exists Q.C \in \mathcal{L}(\mathcal{P}_0)$;*
   *(b) $\mathcal{P}'$ is a $Q$-reachable of $\mathcal{P}$, and $\exists Q.C \in \mathcal{L}(\mathcal{P}')$ where $\mathcal{P} = \mathcal{P}_0$ or $\mathcal{P}$ blocks $\mathcal{P}_0$;*
   *(c) $\mathcal{P}$ is a $Q^{\ominus}$-reachable of $\mathcal{P}'$, and $\exists Q.C \in \mathcal{L}(\mathcal{P}')$ where $\mathcal{P} = \mathcal{P}_0$ or $\mathcal{P}$ blocks $\mathcal{P}_0$.* $\triangleleft$

Conditions 1-3 in Definition 9 ensure satisfaction of tableau properties [14]. In particular, Condition 3 takes into account satisfaction of transitive closure of roles. In fact, for each blocking path $\mathcal{P}$ with $\exists Q^{\oplus}.D \in \mathcal{L}(\mathcal{P})$, this condition says that $\mathcal{P}$ must be satisfied by a $Q$-reachable path $\mathcal{P}'$ of $\mathcal{P}$ between a blocking cycle $\Theta_1$ and a blocked cycle $\Theta_k$. If there is some path $\mathcal{Q}$ between $\Theta_1$ and $\Theta_k$ with $\exists S^{\oplus}.C \in \mathcal{L}(\mathcal{Q})$ that is not satisfied, due to the construction of star-types and frames, a concept $\exists S^{\oplus}.C$ is propagated along $S$-reachable paths of $\mathcal{Q}$. This implies that $\mathcal{Q}$ has a $S$-reachable path $\mathcal{Q}'$ such that $\mathcal{Q}'$ is blocked and $\exists S^{\oplus}.C \in \mathcal{L}(\mathcal{Q}')$. By unravelling (details will be given in soundness proof), we can build an (extended) $S$-reachable path $\mathcal{Q}''$ of $\mathcal{Q}'$ such that $\exists S.C \in \mathcal{L}(\mathcal{Q}'')$.

We now present Algorithm 1 for building a frame which is valid if the conditions in Definition 9 are satisfied. This algorithm starts by adding nominal star-types to the frame. For each non blocked path $\mathcal{P}$ with a ray $\rho \in \xi(\mathsf{tail}_\sigma(\mathcal{P}))$ such that $\delta(\mathsf{tail}_\sigma(\mathcal{P}))$ is minimal and there is a difference between $\delta(\mathsf{tail}_\sigma(\mathcal{P}))$ and $\delta(\mathsf{tail}_\sigma(\mathcal{P}), \rho)$, the algorithm picks in a nondeterministic way a valid star-type $\omega$ that matches $\mathsf{tail}_\sigma(\mathcal{P})$ via $\rho$, and updates $\Omega(\mathsf{tail}_\sigma(\mathcal{P}), \rho)$, $\Omega(\omega, \rho')$, $\delta(\mathsf{tail}_\sigma(\mathcal{P}), \rho)$, $\delta(\omega, \rho')$, eventually, $\delta(\mathsf{tail}_\sigma(\mathcal{P}))$ and $\delta(\omega)$ by calling $\mathsf{updateFrame}(\cdots)$[14]. The algorithm terminates when a blocked cycle is detected.

Figure 2 depicts a frame when executing Algorithm 1 for $\mathcal{K}_1$ in the example presented in Section 1. The algorithm builds a frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ where $\mathcal{N} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and $\mathcal{N}_o = \{\sigma_0\}$. The dashed arrows indicate how the function

**Require:** A $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$
**Ensure:** A frame $\langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ for $(\mathcal{T}, \mathcal{R})$
Let $\Sigma$ be the set of all star-types for $(\mathcal{T}, \mathcal{R})$
**for all** $o \in \mathbf{C}_o$ **do**
  **if** there is no $\sigma \in \mathcal{N}$ such that $o \in \lambda(\sigma)$ **then**
    Choose a star-type $\sigma_o \in \Sigma$ such that $o \in \lambda(\sigma_o)$
    Set $\delta(\sigma_o) = 1, \mathcal{N} = \mathcal{N} \cup \{\sigma_o\}$ and $\mathcal{N}_o = \mathcal{N}_o \cup \{\sigma_o\}$
    Set $\delta(\sigma_o, \rho) = 0$, $\Omega(\sigma_o, \rho) = \emptyset$ for all $\rho \in \xi(\sigma_o)$
  **end if**
**end for**
**while** there is a path $\mathcal{P}$ that is not blocked and a ray $\rho \in \xi(\mathsf{tail}_\sigma(\mathcal{P}))$ such that
    $\mathsf{tail}_\delta(\mathcal{P}) = \delta(\mathsf{tail}_\sigma(\mathcal{P}), \rho) + 1$ and $\delta(\mathsf{tail}_\sigma(\mathcal{P})) \leq \delta(\omega)$ for all $\omega \in \mathcal{N}$ **do**
  Choose a star-type $\sigma' \in \Sigma$ such that there is a ray $\rho' \in \xi(\sigma')$ satisfying
      $l(\rho) = \lambda(\sigma'), l(\rho') = \lambda(\sigma), r(\rho') = r^-(\rho)$, and
      $\sigma' \in \mathcal{N}$ implies $\delta(\sigma') = \delta(\sigma', \rho') + 1$
  $\mathsf{updateFrame}(\sigma, \rho, \sigma', \rho')$
**end while**

**Algorithm 1:** An algorithm for building a frame

$\Omega(\sigma, \rho)$ can be built. For example, $\Omega(\sigma_0, \rho_0) = \{(\sigma_1, \nu_0, 1)\}$, $\Omega(\sigma_0, \rho_1) = \{(\sigma_2, \rho'_0, 1)\}$ where $\rho_0$ and $\rho_1$ are the respective horizontal and vertical rays of $\sigma_0$; $\nu_0$ is the left ray of $\sigma_1$; $\rho'_0$ is the vertical ray of $\sigma_2$. Moreover, the directed dashed arrow from $\sigma_0$ to $\sigma_1$ indicates that the ray $\rho_0$ of $\sigma_0$ can match the ray $\nu_0$ on the left ray of $\sigma_1$ since $l(\rho_0) = \lambda(\sigma_1)$, $r(\nu_0) = \lambda(\sigma_0)$, $r(\nu_0) = r^-(\rho_0)$.

Then, the algorithm generates $\delta(\sigma_0) = 1$, $\delta(\sigma_1) = 1$, $\delta(\sigma_2) = 1$ and forms a cycle $\Theta$ consisting of the following triples : $((\sigma_3, 2), \rho_1, (\sigma_3, 3))$ ($\rho_1$ is the left ray of $\sigma_3$)), $((\sigma_3, 2), \rho_3, (\sigma_4, 1))$ ($\rho_3$ is the vertical ray of $\sigma_3$), $((\sigma_4, 1), \rho_4, (\sigma_4, 2))$ ($\rho_4$ is the left ray of $\sigma_4$) and $((\sigma_4, 1), \rho_5, (\sigma_3, 2))$ ($\rho_5$ is the vertical ray of $\sigma_4$). Note that for the sake of brevity, we use just $\mathsf{tail}_\sigma(\mathcal{P})$ and $\mathsf{tail}_\delta(\mathcal{P})$ to denote a path in the triples. We can check that any path that is an extension of a path $\mathcal{P}$ gets through a path $\mathcal{Q}$ where $\mathcal{P}$ is the first component of a triple and $\mathcal{Q}$ is the third component of a triple.

The algorithm may add some more paths that go through $\sigma_3$ and $\sigma_4$ to form a blocked cycle. A model of the ontology can be built by starting from $\sigma_0$ and getting (i) $\sigma_4$ via $\sigma_1$, (ii) $\sigma_3$ via $\sigma_1$, and (iii) $\sigma_3$ via $\sigma_2$. From $\sigma_3$ and $\sigma_4$, the model goes through $\sigma_3$ and $\sigma_4$ infinitely. Note that from any individual $x$ satisfying $\sigma_3$ (or $\sigma_4$), i.e. the "label" of $x$ contains $\exists Q^+.\{o\}$, there is a path containing $S$ which goes back the individual satisfying $\sigma_0$. Thus, the concept $\exists Q^+.\{o\}$ is satisfied for each individual whose label contains $\exists Q^+.\{o\}$.

**Lemma 1.** *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base.*

1. *Algorithm 1 terminates.*
2. *If Algorithm 1 can build a valid frame for $(\mathcal{T}, \mathcal{R})$ then $(\mathcal{T}, \mathcal{R})$ is consistent.*
3. *If $(\mathcal{T}, \mathcal{R})$ is consistent then Algorithm 1 can build a valid frame $\mathcal{F}$ for $(\mathcal{T}, \mathcal{R})$.*

*Proof (sketch).* Since the functions $\delta(\sigma)$ and $\delta(\sigma, \langle r, l \rangle)$ is increased monotonously by Algorithm 1, termination of the algorithm can be proved if we can show that : (i) the

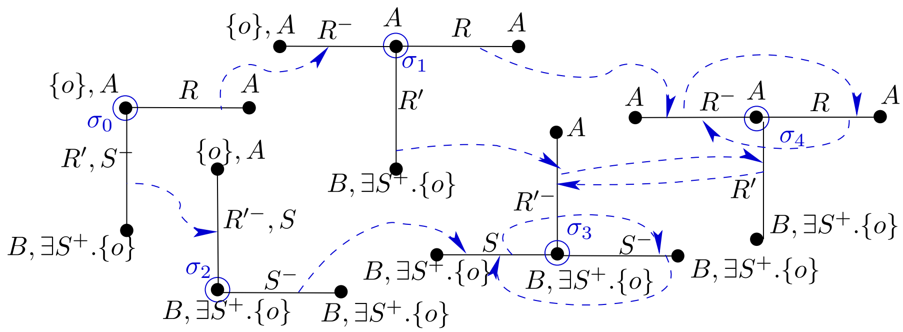


**Fig. 2.** A frame obtained by Algorithm 1 for $\mathcal{K}_1$ in the example in Section 1

number of different star-types is bounded; (ii) the detection of a blocked cycle according to Definition 8 terminates. For the soundness of Algorithm 1, we can extend the set $\mathscr{P}$ of paths to a set $\widehat{\mathscr{P}}$ of extended paths by "unravelling" the frame between blocking-blocked cycles. A tableau [14] can be built from $\widehat{\mathscr{P}}$. The main argument is that when extending a path $\mathcal{P}$ if $\mathsf{tail}_\sigma(\mathcal{P}) \neq \mathsf{tail}_\sigma(\mathcal{P}')$ for all blocked path $\mathcal{P}'$ then this extension process can be continued up to a star-type $\sigma' = \mathsf{tail}_\sigma(\mathcal{P}'')$ for some blocked path $\mathcal{P}''$. This holds due to the definition of cycles and blockable cycles. Otherwise, i.e., $\mathsf{tail}_\sigma(\mathcal{P}) = \mathsf{tail}_\sigma(\mathcal{P}')$ for some blocked path $\mathcal{P}'$ by $\mathcal{Q}$ then $\mathcal{P}$ can be extended by getting through $\mathsf{tail}_\sigma(\mathcal{Q})$.

Regarding completeness, a tableau can guide the algorithm (i) to choose valid star-types, (ii) to ensure that $\delta(\sigma) = 1$ for each nominal star-type $\sigma$, and (iii) to detect a pair $(\Theta_1, \Theta_k)$ of blocking and blocked cycles as soon as each concept of the form $\exists Q^{\oplus}.D$ in $\Theta_1$ is satisfied. We refer the readers to [14] for a complete proof of Lemma 1.

The following theorem is a consequence of Lemma 1.

**Theorem 1.** *$\mathcal{SHOIQ}_{(+)}$ is decidable.*

# 4 Conclusion

In this paper, we have presented a decision procedure for the description logic $\mathcal{SHOIQ}$ with transitive closure of roles in concept axioms, whose decidability was not known. The most significant feature of our contribution is to introduce a structure for characterizing models which have an infinite non-tree-like part. This structure would provide an insight into regularity of such models which would be enjoyed by a more expressive DL, such as $\mathcal{ZOIQ}$ [6], whose decidability remains open. In future work, we aim to improve the algorithm by making it more goal-directed and aim to investigate another open question about the hardness of $\mathcal{SHOIQ}_{(+)}$.

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