Complexity Sources in Fuzzy Description Logic

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Abstract. In recent years many Fuzzy Description Logics (FDLs) based on infinite t-norms have been proved to be undecidable. On the other hand, several FDLs based on finite t-norms, not only have been proved to be decidable, but they have been proved to belong to the same complexity classes as the corresponding crisp DLs. In light of such results, a question that naturally arises is whether the finite-valued fuzzy framework is no more complex than the crisp-valued formalism. The aim of this work is to analyze some of the complexity sources that are not present in the crisp framework. To this end, we will consider FDL languages with low expressivity that allow us to observe how the need for more complex deciding strategies, not required in the crisp framework, arises in many-valued FDLs.

1 Introduction

In recent years many Fuzzy Description Logics (FDLs) based on infinite t-norms have been proved to be undecidable. On the other hand, every FDL based on finite t-norm that has been recently studied, not only has been proved to be decidable, but it has been proved to belong to the same complexity class of the corresponding crisp DL. In light of such results, a question that naturally arises is whether the finite-valued fuzzy framework is no more complex than the crisp-valued formalism. The suspicion is that everything that can be expressed in finite-valued FDLs, can be efficiently reduced to the corresponding crisp DLs.

The aim of this work is to highlight that in the fuzzy framework we have to consider complexity sources that are not present in the crisp framework. In order to analyze this problem, we will consider FDLs where axioms and reasoning tasks use exact values and outline the changes that have to be made in classical tableau-based algorithms in order to cope with such reasoning tasks. In the second part of this work we will give an account on how the fact that Łukasiewicz conjunction is not idempotent impacts on a structural subsumption algorithm for finite Łukasiewicz logic.

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	Minimum (Gödel)	Product (of real numbers)	Łukasiewicz
$x * y$	$\min(x, y)$	$x \cdot y$	$\max(0, x + y - 1)$
$x \Rightarrow y$	$\begin{cases} 1, \text{ if } x \leq y \\ y, \text{ otherwise} \end{cases}$	$\begin{cases} 1, \quad \text{ if } x \leq y \\ y/x, \text{ otherwise} \end{cases}$	$\min(1, 1 - x + y)$
$x \Rightarrow 0$	$\begin{cases} 1, \text{ if } x = 0 \\ 0, \text{ otherwise} \end{cases}$	$\begin{cases} 1, \text{ if } x = 0 \\ 0, \text{ otherwise} \end{cases}$	$1 - x$

Table 1. The three main continuous t-norms.

The present work does not pretend to be an exhaustive account on the complexity sources in many-valued FDLs, but just a starting point towards a systematic study of the complexity issues exclusive to the finite-valued framework. Even though the results presented in this work maintain the same complexity bounds as in the classical cases, they are not a proof that every language whose semantics is based on a finite t-norm is at most as complex as the corresponding classical language.

2 Preliminaries

2.1 Algebras of Truth Values

We will mainly focus our research to the case of finite t-norms.

Definition 1. *A* t-norm *is a binary operation* $*$ *on the real unit interval* $[0, 1]$ *that is associative, commutative, non-decreasing in both arguments and having 1 as neutral (unit) element. The* residuum *of a t-norm $*$ is a binary operation on* $[0, 1]$ *such that* $x * y \leq z \iff x \leq y \Rightarrow z$ *holds for every* $x, y, z \in T$.

Left continuity of $*$, i.e. the property that for any non-decreasing sequence $(x_i)_{i \in I}$ and for any y, $(\bigvee_i x_i) * y = \bigvee_i (x_i * y)$ holds, is a sufficient and necessary condition for the existence of the residuum of the t-norm $*$. A structure $\langle [0, 1], *, \Rightarrow, 0, 1\rangle$, where $*$ is a left-continuous t-norm and $\Rightarrow$ is its residuum, is called *MTL standard chain* (see [9]). This structure is denoted from now on by $[0, 1]_*$. Moreover the same structure, where $*$ is a continuous t-norm and $\Rightarrow$ is its residuum, is called *BL standard chain* (see [11]). The most used representative of the standard chains (unique up to isomorphisms), are the ones defined by the so-called Łukasiewicz, product and minimum t-norms and their residua (collected in Table 1). In the FDL literature, it is natural to restrict to so-called *witnessed models* (see [12]) when the semantics is based on an infinite algebra. Indeed, the main results reported in Section 3 follow such a restriction.

In the case of Łukasiewicz and Gödel t-norms $*$, the operations in Table 1 can be defined on the domain of a finite subalgebra of $[0, 1]_*$. In these cases, we can talk about *finite t-norms*. So, for every natural number n, $\mathbf{Ł_n}$ and $\mathbf{G_n}$ will denote the restriction of Łukasiewicz and Gödel t-norms, respectively, to the subalgebra of cardinality n over the domain $T = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ in the n-valued case. Notice that finite product subalgebras of $[0, 1]_\Pi$ (of cardinality > 2) do not exist, so a finite product t-norm can not be defined.

As it has been proved in [16], every t-norm is definable as *ordinal sums* of the basics continuous t-norms from Table 1. The result in [16] is the corresponding result for t-norms of the one that in [17] has been proved for families of abelian semi-groups. A simple corollary, about the restriction of this result to finite t-norms is the following Proposition.

Proposition 2. *Every finite BL-chain is an ordinal sum of isomorphic copies of Łukasiewicz and Gödel finite chains.*

Relying on the result of Proposition 2, we will restrict our study to the case of Łukasiewicz and Gödel finite chains.

2.2 Syntax

A *description signature* is a tuple $\mathcal{D} = \langle N_I, N_C, N_R \rangle$, where $N_I = \{a, b, \dots\}$ is a countable set of *individual names*, $N_C = \{A, B, \dots\}$ is a countable set of *atomic concepts* or *concept names* and $N_R = \{R, S, \dots\}$ is a countable set of *atomic roles* or *role names*. *Complex concepts* in the FDL languages considered in this work are built inductively from atomic concepts and roles by means of the corresponding subset of the following concept constructors:

C, D	$\longrightarrow$	A	atomic concept	$\mathcal{FL}_0$
		$C \sqcap D$	strong conjunction	$\mathcal{FL}_0$
		$\forall R.C$	value restriction	$\mathcal{FL}_0$
		$\exists R.\top$	restricted existential quantif.	$\mathcal{FL}^-$
		$\sim A$	atomic complementation	$\mathcal{AL}$
		$\sim C$	complementation	$\mathcal{C}$
		$C \sqcup D$	strong disjunction	$\mathcal{U}$
		$\exists R.C$	existential quantification	$\mathcal{E}$
		$C \to D$	implication	$\mathfrak{I}$

where $A \in N_C$, and $R \in N_R$. In the rest of the paper we will refer to the right column of the above table to denote the constructors explicitly present in each language considered. The rules for reading it are the usual one in the framework of DL (see [1]).

2.3 Semantics

Given a finite chain $\mathbf{T} = \langle T, *, \Rightarrow, 0, 1 \rangle$ an interpretation is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a nonempty (crisp) set $\Delta^{\mathcal{I}}$ (called *domain*) and of a *fuzzy interpretation function* $\cdot^{\mathcal{I}}$ that assigns a fuzzy set $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \longrightarrow T$ to each concept name $A \in N_C$, a fuzzy relation $R^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \longrightarrow T$ to each role name $R \in N_R$ and an object $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ to each individual name $a \in N_I$. The semantics of complex concepts is a function $C^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \longrightarrow T$ inductively defined as follows:

$$\begin{array}{rcl}
(\sim C)^{\mathcal{I}}(v) & := & 1 - C^{\mathcal{I}}(v) \\
(C \sqcap D)^{\mathcal{I}}(v) & := & C^{\mathcal{I}}(v) * D^{\mathcal{I}}(v) \\
(C \sqcup D)^{\mathcal{I}}(v) & := & 1 - ((1 - C^{\mathcal{I}}(v)) * (1 - D^{\mathcal{I}}(v))) \\
(C \to D)^{\mathcal{I}}(v) & := & C^{\mathcal{I}}(v) \Rightarrow D^{\mathcal{I}}(v) \\
(\forall R.C)^{\mathcal{I}}(v) & := & \inf_{w \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(v, w) \Rightarrow C^{\mathcal{I}}(w)\} \\
(\exists R.C)^{\mathcal{I}}(v) & := & \sup_{w \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(v, w) * C^{\mathcal{I}}(w)\}
\end{array}$$

2.4 Axioms and Reasoning Tasks

In the fuzzy framework, axioms are not necessarily asked to take value 1. Rather they can be explicitly asked to take other values. In the literature (see [3]), there are publications where concept inclusions and assertions are allowed to take a whole set of truth values. Usually, the sets of truth values considered are included between a positive value $r > 0$ and 1, that is, the graded axioms have the following form:

$$\langle C \sqsubseteq D \geq r \rangle, \tag{1}$$

$$\langle C(a) \geq r \rangle. \tag{2}$$

Moreover, assertion axioms can be asked to take single values only, different than 1 (see [10]), then having the following form:

$$\langle C(a) = r \rangle. \tag{3}$$

Sometimes exact value inclusions have been addressed in the literature, mainly as mathematical problems (for example in [10]). Indeed, the conditions required by these kinds of axioms are quite anti-intuitive w.r.t. the expected behavior of an inclusion axiom. For this reason we are not considering here.

Besides graded axioms, also graded notions of reasoning tasks like satisfiability and subsumption can be considered. Again, these reasoning tasks can be considered either w.r.t. a lower bound (see [21] for an overview), or w.r.t. exact values (see [6,12]). Speaking about satisfiability, in the first case the question is whether there is a model $\mathcal{I}$ of a (possibly empty) KB, which satisfies a concept C to a certain degree r' between a value r and 1. In the second case the question is whether there is a model $\mathcal{I}$ of a (possibly empty) KB, which satisfies a concept C to a certain degree r. Asking whether concept C is subsumed by concept D w.r.t a (possibly empty) KB to a certain degree r means asking whether, for every element $x \in \Delta^{\mathcal{I}}$ in every model $\mathcal{I}$ of KB, the implication $C^{\mathcal{I}}(x) \rightarrow D^{\mathcal{I}}(x)$ always takes a value greater or equal than r. Subsumption w.r.t. an exact value is not considered for the same reason as for exact value inclusions above.

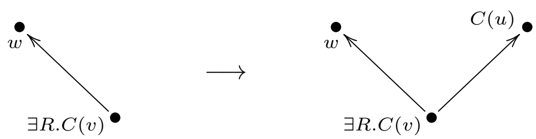
3 Lower Bounds vs Exact Values

Intuitively, the information brought by an axiom with an exact value, like (3), can be equivalently expressed by means of a lower bound axiom, like (2), plus a suitable upper bound axiom. For this reason, an exact value axiom is stronger than a lower bound axiom, since it brings with it more information. This increment on the strength side is reflected on increased computational costs.

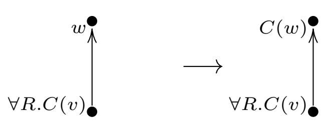
Some consequences of these higher costs have been already studied in the infinite-valued case. In [5] it is proved, among other things, that language $[0,1]_*$-$\mathfrak{I}\mathcal{ALE}$ (without atomic complementation) with lower bound axioms is undecidable when the algebra of truth values $[0,1]_*$ is a t-norm whose ordinal sum begins with a Łukasiewicz chain. But, if exact value axioms are allowed, language $[0,1]_*$-$\mathfrak{I}\mathcal{ALE}$ becomes undecidable with every infinite t-norm $[0,1]_*$ but

Gödel as algebra of truth values. This result is enhanced by the results in [3] and [2]. In [3] it is proved that, when just lower bound axioms are allowed, language Π-$\mathcal{SHOI}$ without complementation is linearly reducible to classical $\mathcal{SHOI}$ and, therefore, its computational complexity is the same as in the classical case. In [2] on the other hand, it is proved that the simpler Π-$\mathcal{ALE}$ becomes undecidable if assertions with exact value are allowed.

Tableau-like algorithms for lower bound reasoning tasks. But even though the presence of exact value axioms does not lead to undecidability in the case of FDLs valued on a finite algebra, it indeed leads to an increase of the computational costs. In order to understand how these computational costs increase, we have to recall the basic completion rules for fresh constraint trees defined in [19]. There, a tableau-based algorithm for concept satisfiability in classical $\mathcal{ALC}$ has been defined. The same algorithm has been subsequently used, expanded for more expressive languages and reasoning tasks (see [14]) and generalized to the fuzzy framework (see [8]). As we can see in [19], the classical tableau algorithm, while building the tableau interpretation, adds a new element every time it finds out an existentially quantified subconcept $\exists R.C$ and subsequently, it assigns concept C to the newly added element.

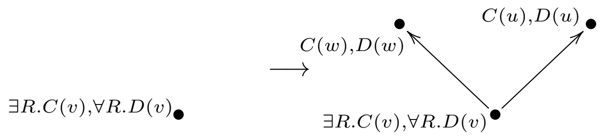


On the other hand, when a value restriction $\forall R.C$ is found, no new element is added to the tableau interpretation under construction, but concept C is assigned to the already existing elements, if any.



As we can see for example in [8], the same procedure can be used for solving acyclic knowledge base consistency in the context of language $\text{Ł}_\mathbf{n}$-$\mathcal{ALC}$, when just lower bound axioms are allowed.

Tableau-like algorithms for exact value reasoning tasks. Nevertheless, when exact value axioms are allowed, this procedure can not be used any longer. In [12] algorithms for concept satisfiability (without TBox) w.r.t. an exact value are provided for languages Ł-$\mathcal{ALC}$. The algorithm provided in these publication is based on a reduction to the corresponding propositional logic and is very similar to the tableaux algorithm provided in [8]. The main difference is that this algorithm works by adding a new element not only when an existentially quantified subconcept $\exists R.C$ is found, but also when a value restriction $\forall R.C$ has to be computed.



The algorithms provided in [12] is still in the context of infinite-valued FDLs. But similar procedures are also used in [6] and [4] in the context of finite-valued FDLs. In [6] an algorithm for solving concept satisfiability to an exact value in $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCE}$ w.r.t. empty knowledge bases is provided, where $\mathbf{T}$ stands for any finite t-norm. In [4] an algorithm for solving knowledge base consistency to an exact value in language $\mathbf{T}$-$\mathcal{SHI}$ w.r.t. local ABoxes is provided, where $\mathbf{T}$ stands for any finite residuated De Morgan lattice. Again, the interesting feature of the procedures provided in [6] and [4] is the fact that they add a new element to the tableaux-like structure both when an existential quantification $\exists R.C$ or a value restriction $\forall R.C$ are found. As proved in the cited publications, these different algorithms do not increase the computational complexity classes their languages belong to, w.r.t. the classes of the classical DL equivalent languages or to the classes of the corresponding FDL languages where just either lower bound axioms or lower bound reasoning tasks are considered. Nevertheless it translates into greater computational costs, in the sense that the size of the model to be built is greater. This means that the algorithm builds a greater propositional theory in the case of [6] and a greater constraint set in the case of [4].

Finite Łukasiewicz chains. In the case of FDLs based on finite Łukasiewicz chains, the explanation is quite simple. Due to the properties of finite Łukasiewicz chains, in fact, we have that, for every interpretation $\mathcal{I}$, $v \in \Delta^{\mathcal{I}}$ and $r \in \text{Ł}_n$:

$$C^{\mathcal{I}}(v) = r \quad \text{iff} \quad \text{both } C^{\mathcal{I}}(v) \geq r \text{ and } \neg C^{\mathcal{I}}(v) \geq 1 - r.$$

In particular, for the case of a value restriction $\forall R.C$, we have that

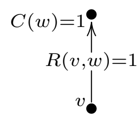
$$\forall R.C^{\mathcal{I}}(v) = r \quad \text{iff} \quad \text{both } \forall R.C^{\mathcal{I}}(v) \geq r \text{ and } \exists R.\neg C^{\mathcal{I}}(v) \geq 1 - r.$$

That is, for deciding exact value reasoning tasks in $\mathbf{Ł_n}$ would be enough to use the simpler classical-like algorithm, but behind the satisfiability of a value restriction to an exact value, there is the lower bound satisfiability of an existentially quantified concept hidden. And for each existentially quantified concept a new element must be added, according to every kind of tableau algorithm.

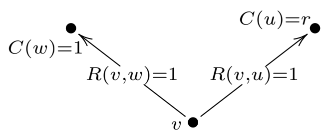
Finite Gödel chains. In the case of finite Gödel chains $\mathbf{G_n}$, value restrictions and existentially quantified concepts are not interdefinable by means of the negation. Nevertheless, here we want to prove, by means of a counter-example, that the classical-like algorithms can not be used in presence of exact value reasoning tasks. Indeed, consider $r \in G_n$ such that $0 < r < 1$ and the following concept:

$$\exists R.C \to \forall R.C. \tag{4}$$

That concept (4) is satisfiable at least to a value r means that there exist an interpretation $\mathcal{I}$ and $v \in \Delta^{\mathcal{I}}$, such that $(\exists R.C \to \forall R.C)^{\mathcal{I}}(v) \geq r$. For example, the $\mathbf{G_n}$-interpretation $\mathcal{I}_1$:



satisfies (4) since $(\exists R.C \to \forall R.C)^{\mathcal{I}_1}(v) = 1 \geq r$. But concept (4) is not satisfied in any $x \in \Delta^{\mathcal{I}_1}$ to an exact value $r < 1$. Nevertheless, concept (4) is indeed satisfiable at point v in the $\mathbf{G_n}$-interpretation $\mathcal{I}_2$:



It can indeed be easily proved that concept (4) can not be satisfied to an exact value $0 < r < 1$ in any point v of a $\mathbf{G_n}$-interpretation $\mathcal{I}$ with less than two R-successors.

Proposition 3. *If concept (4) is satisfied to an exact value* $0 < r < 1$ *in a point* v *of a* $\mathbf{G_n}$*-interpretation* $\mathcal{I}$*, then* v *has at least two* R*-successors.*

Proof. Suppose, in search of a contradiction, that there exist an interpretation $\mathcal{I}$ and $v \in \Delta^{\mathcal{I}}$ such that $(\exists R.C \to \forall R.C)^{\mathcal{I}}(v) = r$, but v has less than two R-successors. If v has no R-successors, then it is obvious that $(\exists R.C \to \forall R.C)^{\mathcal{I}}(v) = 1 > r$, so let us see the case when v has just one R-successor.

Let w be the unique R-successor of v. Since $(\exists R.C \to \forall R.C)^{\mathcal{I}}(v) = r < 1$, then $s = (\exists R.C)^{\mathcal{I}}(v) > (\forall R.C)^{\mathcal{I}}(v) = r$. Therefore both $R^{\mathcal{I}}(v,w) \wedge C^{\mathcal{I}}(w) = s$ and $R^{\mathcal{I}}(v,w) \Rightarrow_{\mathbf{G_n}} C^{\mathcal{I}}(w) = r$. Since $R^{\mathcal{I}}(v,w) \Rightarrow_{\mathbf{G_n}} C^{\mathcal{I}}(w) = r < 1$, then we have that $R^{\mathcal{I}}(v,w) > C^{\mathcal{I}}(w) = r$. Therefore $R^{\mathcal{I}}(v,w) \wedge C^{\mathcal{I}}(w) = r < s = R^{\mathcal{I}}(v,w) \wedge C^{\mathcal{I}}(w)$, a contradiction. Hence, any point v of a $\mathbf{G_n}$-interpretation $\mathcal{I}$ which satisfies concept (4) to an exact value $0 < r < 1$ must have at least two R-successors. □

4 Structural Subsumption Algorithms for Many-valued FDLs

We address now the possibility of generalizing the structural subsumption algorithm for the classical description language $\mathcal{FL}^-$. This language is interesting for us because, as it has been proved in [7], it has a polynomial time subsumption problem. Moreover, the problem of generalizing structural subsumption algorithms to the many-valued framework, as far as we know, until now has been addressed only in [22] under a semantics different from t-norms, but it has not still been faced with this kind of semantics. In [7] a structural subsumption algorithm $SUBS?[a,b]$ for deciding concept subsumption in $\mathcal{FL}^-$ is presented. The fact that $SUBS?[a,b]$ calculates subsumption in $\mathcal{O}(n^2)$ relies on the fact that every $\mathcal{FL}^-$ concept C is equivalent to a $\mathcal{FL}^-$ concept C^* where each value restriction $\forall R.$ appears at most once for each nesting level. That is, for example concepts $\forall R.(C \sqcap D)$ and $\forall R.C \sqcap \forall R.D$ are equivalent.

The structural subsumption algorithm $SUBS?[a,b]$ can be indeed consistently used in order to decide 1-subsumption[3] for $\mathbf{G_n}$-$\mathcal{FL}^-$. This is due to the fact that the Gödel t-norm $\wedge$ works well with its residuum $\Rightarrow_{\mathbf{G_n}}$. That is, for every $x, y, z \in G_n$:

$$x \Rightarrow_{\mathbf{G_n}} (y \wedge z) = (x \Rightarrow_{\mathbf{G_n}} y) \wedge (x \Rightarrow_{\mathbf{G_n}} z) \ . \tag{5}$$

Hence, correctness of algorithm $SUBS?[a,b]$ w.r.t. 1-subsumption problem for $\mathbf{G_n}$-$\mathcal{FL}^-$ is due to the fact that $(\varphi \& \psi) \to \varphi$ is a theorem of Gödel logic, while completeness can be easily deduced from the fact that, if $SUBS?[a,b]$ returns a negative answer, then a counter-example to 1-subsumption can be easily found.

Unfortunately, the same result does not hold between Łukasiewicz t-norm $*_{Ł_n}$ and its residuum $\Rightarrow_{Ł_n}$. That is, there are $x, y, z \in Ł_n$ such that

$$x \Rightarrow_{Ł_n} (y *_{Ł_n} z) \neq (x \Rightarrow_{Ł_n} y) *_{Ł_n} (x \Rightarrow_{Ł_n} z) \ . \tag{6}$$

As an example, if we take $x = y = z = 0.8$, then we have that $x \Rightarrow_{Ł_n} (y *_{Ł_n} z) = 0.8 \neq 1 = (x \Rightarrow_{Ł_n} y) *_{Ł_n} (x \Rightarrow_{Ł_n} z)$. Since the residuum plays a fundamental role in the semantics of value restrictions in FDL, we have as a consequence that in $\mathbf{Ł_n}$-$\mathcal{FL}^-$, concepts $\forall R.(C \sqcap D)$ and $\forall R.C \sqcap \forall R.D$ are not equivalent. Nevertheless, the overall complexity of the subsumption problem does not increase. In order to see it, we will consider separately 1-subsumption and $(\geq r)$-subsumption for $r \in (0,1)$. As we will see at the end of this section the notion of $(= r)$-subsumption for $r \in (0,1)$ does not make sense in $\mathbf{Ł_n}$-$\mathcal{FL}^-$.

1-subsumption in $\mathbf{Ł_n}$-$\mathcal{FL}^-$. The possibility of applying structural algorithms to a given calculus is due to the fact that concept conjunctions can be considered as sets of concepts. Since Łukasiewicz conjunction is not idempotent, complex concepts where just $\sqcap$ appears as concept constructor can not be seen as sets of atomic concepts. In this sense, an inclusion like $A \sqsubseteq A \sqcap A$ which is valid in classical $\mathcal{FL}^-$ or in $\mathbf{G_n}$-$\mathcal{FL}^-$, is not a 1-subsumption in $\mathbf{Ł_n}$-$\mathcal{FL}^-$. Nevertheless, complex concepts in $\mathbf{Ł_n}$-$\mathcal{FL}^-$ can be seen as *multisets* of simpler concepts, that is, different *occurrences* of atomic concepts are now seen as different elements of a given complex concept.[4] This gives us the possibility of still defining structural subsumption algorithms for $\mathbf{Ł_n}$-$\mathcal{FL}^-$, as showed in the following proposition.

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[3] Note that subsumption between two concepts in $\mathbf{G_n}$-$\mathcal{FL}^-$ always takes either value 0 or value 1. Therefore, speaking about $(\geq r)$- or $(= r)$-subsumption in $\mathbf{G_n}$-$\mathcal{FL}^-$ does not make sense.

[4] In what follows, when not otherwise stated, restricted existential quantifications will be treated as atomic concepts, with the assumption that two concepts $\exists R.\top$ and $\exists S.\top$ are different occurrences of the same concept iff $R = S$.

Proposition 4. *Let C, D be complex $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$ concepts where just conjunction $\sqcap$ and restricted existential quantification $\exists R.\top$ appear. Then C is subsumed by D iff every occurrence of a conjunct A that appears in D, also appears in C, where A appears in C strictly less than $n-1$ times.*

Proof. Let C, D be complex concepts where just conjunction $\sqcap$ and existential quantification $\exists R.\top$ appear. The right to left direction is trivial due to the fact that $(\varphi \& \psi) \to \varphi$ is a theorem of $Ł_{\mathbf{n}}$ and, for every propositional evaluation e, it holds that $e(\varphi \& \overset{n-1}{\ldots} \& \varphi) \in \{0, 1\}$. The left to right direction can be proved by contradiction. Suppose that there exists an occurrence of a conjunct A that appears in D but not in C, where A appears in C strictly less than $n-1$ times. We have two cases: either A is an atomic concept or $A = \exists R.\top$. In the first case an interpretation $\mathcal{I}$ where $\Delta^{\mathcal{I}} = \{v\}$, $A^{\mathcal{I}}(v) = \frac{n-2}{n-1}$ and $B^{\mathcal{I}}(v) = 1$ for $B \neq A$ gives us an example against subsumption of C by D. In the second case, consider an interpretation $\mathcal{I}$ where $\Delta^{\mathcal{I}} = \{v, w\}$, $R^{\mathcal{I}}(v, w) = \frac{n-2}{n-1}$, and $S^{\mathcal{I}}(v, w) = B^{\mathcal{I}}(v) = 1$ for every atomic concept B and every role name S different from R. Then $C^{\mathcal{I}}(v) > D^{\mathcal{I}}(v)$ and, therefore, $\mathcal{I}$ is a counterexample against subsumption of C by D. $\square$

This is true up to conjunctions out of the scope of any quantifier. To see how does it work in the case of more complex concepts, recall that the semantics of value restrictions is defined by means of the residuum $\Rightarrow_{Ł_n}$ of the Łukasiewicz t-norm. This operation is monotonic in the second argument which is again a complex conjunction. So, from Proposition 4 we easily obtain Corollary 5.

Corollary 5. *Let $\forall R.C, \forall R.D$ be $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$ concepts. Then $\forall R.C$ is subsumed by $\forall R.D$ iff every occurrence of an atomic concept A in D, also is in C, where the number of occurrences of A in C is strictly less than $n-1$, and every occurrence of a value restriction $\forall S.E$ in C is subsumed by a respectively different occurrence of a value restriction $\forall S.F$ in D.*

From these results, a polynomial time algorithm for deciding 1-subsumption in $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$ can be obtained. The following Algorithm 1 is an adaption of the procedure for solving tree inclusion presented in [18] to the FDL case. Soundness and completeness of Algorithm 1 follow from Proposition 4 and Corollary 5. On the other hand, the fact that Algorithm 1 is polynomial can be showed as in [18]. Indeed, steps 1 and 5 can be performed in linear time and each matrix $\mathbf{E}_{E,F}$ is at most quadratic on the size of the largest concept between C and D. Please note that $\mathbf{E}_{E,F}$ contains a row for each value restriction $\forall R.F$ which is a conjunct of D and, specifically, if there are multiple occurrences of the same expression then there are multiple rows for them. The same observation applies for the columns in $\mathbf{E}_{E,F}$. Moreover, there are at most $|C| \times |D|$ different matrices $\mathbf{E}_{E,F}$. The only non-deterministic problem is to decide whether every value restriction $\forall R.F$ which is a conjunct of a given subconcept D' of D is subsumed by a different value restriction $\forall R.E$ which is a conjunct of a given subconcept C' of C. But, as in [18], instead of checking out this fact for different non-deterministic guesses, a suitable procedure for the bipartite matching problem (see [13] for example) can give an answer in polynomial time.

Algorithm 1 $\text{Ł}_{\mathbf{n}}$-$SUBS(1, D, C)$

1: **if** there is an occurrence of an atomic or existential conjunct A of D that is not in C where concept A appears in C strictly less $n-1$ times **then return** 0
2: **else**
3: $\quad \mathbf{E}_{C,D} := \emptyset$
4: $\quad$ **for all** value restriction $\forall R.F$ which is a conjunct of D **do**
5: $\quad\quad$ **for all** value restriction $\forall R.E$ which is a conjunct of C **do**
6: $\quad\quad\quad \mathbf{E}_{C,D}(\forall R.F, \forall R.E) := \text{Ł}_{\mathbf{n}}\text{-}SUBS(1, F, E)$
7: $\quad\quad$ **end for**
8: $\quad$ **end for**
9: $\quad$ **if** there is a maximal bipartite matching for $\mathbf{E}_{C,D}$ **then return** 1
10: $\quad$ **else**
$\quad\quad$ **return** 0
11: $\quad$ **end if**
12: **end if**

($\geq r$)-subsumption in $\text{Ł}_{\mathbf{n}}$-$\mathcal{FL}^-$. Saying that an $\text{Ł}_{\mathbf{n}}$-$\mathcal{FL}^-$ concept C is ($\geq r$)-subsumed by another $\text{Ł}_{\mathbf{n}}$-$\mathcal{FL}^-$ concept D in degree greater or equal than r means that in every interpretation $\mathcal{I}$ and for every $v \in \Delta^{\mathcal{I}}$ we have that $C^{\mathcal{I}}(v) \Rightarrow D^{\mathcal{I}}(v) \geq r$. In order to design a suitable modification of Algorithm 1 that decides ($\geq r$)-subsumption in $\text{Ł}_{\mathbf{n}}$-$\mathcal{FL}^-$ it is worth recalling that under Łukasiewicz semantics, for every propositional formula φ and every propositional evaluation e, it holds that:

$$e((\varphi \& \overset{i}{\ldots} \& \varphi) \to (\varphi \& \overset{j}{\ldots} \& \varphi)) \in [\min\{1, \frac{i}{j}\}, 1] \cap L_n \tag{7}$$

That is, the value of formula φ in every propositional model, is constrained by the relation between the respective occurrences of p in both sides of the implication. Now, consider a formula of the form of (7) where $\varphi \in \{p, q\}$, that is

$$(p \& \overset{i}{\ldots} \& p \& q \& \overset{k}{\ldots} \& q) \to (p \& \overset{j}{\ldots} \& p \& q \& \overset{l}{\ldots} \& q) \tag{8}$$

Suppose, without loss of generality, that $\min\{\frac{i}{j}, \frac{k}{l}\} = \frac{i}{j}$. Since $\min\{\frac{i}{j}, \frac{k}{l}\} \leq \frac{i+k}{j+l}$, then the minimum assignment for (8) is obtained by a propositional evaluation e that minimizes $e((p \& \overset{i}{\ldots} \& p) \to (p \& \overset{j}{\ldots} \& p))$ and assigns value 1 to variable q. For the same reason, this evaluation is minimal with each number of propositional variables. In this way, we can perform a first modification of Algorithm 1 by assigning a weight in step 5 to the couples in the set $\mathbf{E}$. A further modification of Algorithm 1 consists in substituting, in step 8, a procedure for the *assignment problem* (see for example [15]) instead of the one for bipartite matching. Once obtained a maximal weighted matching, there should be checked that the Łukasiewicz conjunction of the weights is still greater or equal than value r. All the steps of the modified algorithm are still polynomial. Indeed assigning a weight to a couple E, F of concepts is just a matter of counting the occurrences of the same quantified or atomic concepts in E and F respectively. The assignment problem is known to be polynomial and Łukasiewicz t-norm operation is polynomial on fixed values. This proves that $(\geq r)$-subsumption in $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$ is still polynomial.

$(= r)$-subsumption in $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$. Finally we address the $(= r)$-subsumption problem in $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$. Indeed, if $r \in (0, 1)$, then there is no couple C, D of $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$ concepts such that C is $(= r)$-subsumed by D. In order to see this, for each pair C, D of $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$ concepts, consider the interpretation $\mathcal{I}_{C,D}$ defined by:

- $\Delta^{\mathcal{I}_{C,D}} = \{v_1, \ldots, v_{\max\{deg(C), deg(D)\}}\}$, being $deg(C)$ the maximal nesting degree of concept C,
- $A^{\mathcal{I}_{C,D}}(v) = 1$, for every $v \in \Delta^{\mathcal{I}_{C,D}}$ and every atomic concept A in C or D,
- $R^{\mathcal{I}_{C,D}}(v, w) = 1$, for every $v, w \in \Delta^{\mathcal{I}_{C,D}}$ and every atomic role R in C or D.

It is clear that such an interpretation exists for each pair C, D of $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$ concepts and that $C^{\mathcal{I}_{C,D}}(v) = 1$ and $D^{\mathcal{I}_{C,D}}(v) = 1$ for every $v \in \Delta^{\mathcal{I}_{C,D}}$. Hence it is impossible that an $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$ concept C is $(= r)$-subsumed by another $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$ concept D in degree equal to r, as

$$\inf_{w \in \Delta^{\mathcal{I}_{C,D}}} \{C^{\mathcal{I}_{C,D}}(w) \Rightarrow D^{\mathcal{I}_{C,D}}(w)\} = 1 > r \tag{9}$$

So, $\mathcal{I}_{C,D}$ is a counter-example against the existence of $(= r)$-subsumptions in $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$.

5 Conclusions

In this paper we have tried to give an account on some complexity sources that, additionally to the ones that are inherited from classical Description Logics, are proper of the many-valued framework. In particular we have analyzed the additional work that is required from a procedure that is asked to solve reasoning tasks that are only definable in presence of multiple truth values and the effects of the fact that Łukasiewicz conjunction is not idempotent. In the first case, in Section 3 we have summarized the results existing in the literature. Moreover we have tried to explain by means of examples why algorithms for deciding reasoning tasks related to an exact value appear to be more complex than those algorithms for the same reasoning tasks related to a lower bound or to the crisp case. In the second case, we have described a couple of non trivial modifications of the classical structural subsumption algorithm that solve the 1-subsumption and the lower bound subsumption problems respectively for the language $\mathcal{FL}^-$ with semantics based on finite Łukasiewicz t-norm. These enhanced procedures show that subsumption in $Ł_{\mathbf{n}}$-$\mathcal{FL}^-$ is still polynomial. As future work we plan to see whether the same modifications to the structural subsumption algorithms that we provide here, can be extended to other languages that, in the classical case also use these kinds of algorithms. An example of these languages is the one presented in [20] about $\mathcal{ALN}$. It would be also interesting to study how the behavior of structural subsumption algorithms generalizes to the case of more general finite t-norms, from the base cases studied here. Another direction for future works is a more systematic settlement of the method sketched in the present work for outlining complexity shifts in the many-valued case.

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